Seesaw Mechanism with Occam's Razor

- A fast track towards the spacetime geometry? -

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Based on Phys.Rev.D86(2012)013002 (K.Harigaya, MI, T.T.Yanagida)

Introduction

- Why are we so much interested in neutrino mass?

- Neutrino mass is a window to high energy physics beyond the Standard Model !

- Tiny ! → New mass scales ? New symmetry ?
- Mixing ! → Implications on flavor structure ?
- Majorana ? → Lepton number violation ?
- CP-violation ? → Baryon asymmetry of the universe ?

Introduction

Seesaw Mechanism ['79 Yanagida; '79 Gell-Mann, Ramond, Slansky]

In the Standard Model :

$$\mathcal{L} = y_{\alpha\beta}\ell_{L\alpha}\bar{e}_{R\beta}h \qquad \langle h \rangle = v \simeq 174.1\,\mathrm{GeV}$$

(α, β =e, μ, τ) → the neutrinos remain massless !

Let us introduce the right-handed neutrinos (N_i) :

$$\mathcal{L} = y_{\alpha\beta}\ell_{L\alpha}\bar{e}_{R\beta}h + \lambda_{i\alpha}N_i\ell_{L\alpha}h - \frac{1}{2}M_{ij}N_iN_j$$

$$\rightarrow \mathcal{L}^{\nu}_{\mathrm{mass}} = -\frac{1}{2}\left[(\nu_L, N_R)\begin{pmatrix} 0 & m_D^T \\ m_D & M \end{pmatrix}\begin{pmatrix} \nu_L \\ N_R \end{pmatrix}\right] + h.c.$$

→ the neutrinos have finite masses : $m_\nu \simeq \dfrac{m_D m_D^T}{M}$

$m_\nu = O(0.01)$ eV for $M = O(10^{11})$ GeV & $m_D = O(1)$GeV !

Introduction

✓ Leptogenesis [`86 Fukugita & Yanagida]

Baryon asymmetry (from nucleosynthesis and CMB):

$$\eta_{B_0} = \frac{n_B - n_{\bar{B}}}{n_\gamma} \simeq 6 \times 10^{-10}$$

must have been generated during the evolution of the universe.

Sakharov three conditions ('67) :

- ✓ *B* (or *B-L*) symmetry breaking
- ✓ *C* and *CP* violation
- ✓ *B-L* and *C/CP* violating interactions outside of thermal equilibrium

Introduction

✔ Leptogenesis [`86 Fukugita & Yanagida]

Time ↓

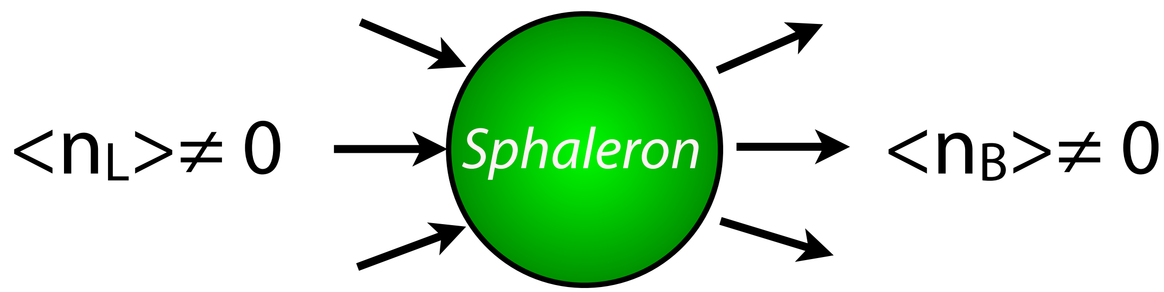
Inflation : $T \to 0$, $\eta_{B0} \to 0$

Reheating : $T \to T_R$, $\eta_{B0} = 0$

N_R are in the thermal equilibrium ($T \gg M_R$)

N_R decays at a temperature T_D

Lepton asymmetry is generated !



Baryon asymmetry is generated !

Sakharov conditions

1: N_R mass violates L

$$N_R \to \ell + h,\ \ell^* + h^*$$

2: CP-violating decay

$$\Gamma[N_R \to \ell + h] \neq \Gamma[N_R \to \ell^* + h^*]$$

3: Out of equillibrium

$$M_R/T_D \gtrsim 1$$

$$\eta_{B0} \simeq 3 \times 10^{-10} \times \left(\frac{M_R}{10^{10}\,\mathrm{GeV}}\right)\left(\frac{m_\nu^{\mathrm{eff}}}{0.05\mathrm{eV}}\right)\bar{\kappa}\sin\delta_{\mathrm{eff}}$$

$$\bar{\kappa} \simeq \left(\frac{0.01\,\mathrm{eV}}{\tilde{m}_1}\right)^{1.16} \quad \left[\ \tilde{m}_1 = \sum_\alpha |\lambda_{1\alpha}|^2 \frac{v^2}{M_R} \propto \frac{T_D^2}{M_R^2}\ \right]$$

In the seesaw mechanism...

- ✓ Tiny neutrino mass can be explained by a new scale
  = Right handed neutrino mass !

- ✓ With the *CP*-violating phases in the right-handed neutrino sector, the *Baryon Asymmetry* of the universe can be explained by *Leptogenesis*.

Future observations of the *CP*-asymmetry in the neutrino oscillations and the neutrino-less double beta decay, will support the ideas of the *seesaw mechanism* and *Leptogenesis* qualitatively.

- ✓ To what extent will we learn the seesaw mechanism and Leptogenesis quantitatively?

Seesaw Mechanism vs Neutrino oscillation

Number of real valued parameters

Seesaw Mechanism			Low energy theory	
M_i	3	>	M_i	3
$y_{\alpha\beta}$	3		$y_{\alpha\beta}$	3
$\lambda_{i\alpha}$	15 = (18-3)		$\bar{m}_{\nu i}$	3
			U_{MNS}	6 = 3 + 1 + 2

[Mass diagonalized base]

For given $\bar{m}_{\nu i}$ and U_{MNS} in the seesaw mechanism

$$\bar{m}_\nu \;=\; U_{MNS}^T \lambda^T M_R^{-1} \lambda U_{MNS} v^2$$

the Yukawa coupling λ is determined up to R,

$$\lambda = \frac{1}{v} M_R^{1/2} R \bar{m}_\nu^{1/2} U_{MNS}^\dagger$$

which satisfies $R^T R = 1$ (i.e. complex orthogonal matrix = 6 parameters).

The Yukawa coupling λ cannot be determined by the low energy data...

Seesaw Mechanism vs Neutrino oscillation

Relation between *CP*-violating phases :

Neutrino oscillation : Dirac *CP*-phase δ in U_{MNS}

$$A_{CP} = P(\nu_l \to \nu_{l'}) - P(\bar{\nu}_l \to \bar{\nu}_{l'}) \propto J_{CP} = Im[U_{\mu 3} U_{e3}^* U_{e2} U_{\mu 2}^*]$$
$$= (sin2\theta_{12}\, sin2\theta_{12}\, sin2\theta_{13}\, cos\theta_{13} sin\delta)/8$$

Leptogenesis : *CP*-phase of the redundant parameters in R

$$\eta_{B0} \propto m_\nu^{\rm eff} \sin \delta_{\rm eff} \qquad m_\nu^{\rm eff} \sin \delta_{\rm eff} = \frac{{\rm Im}[\lambda m_\nu \lambda^T]_{11}}{(\lambda\lambda^\dagger)_{11}}$$

$$\lambda\lambda^\dagger = \frac{1}{v^2} M_R^{1/2} R \bar{m}_\nu R^\dagger M_R^{1/2} \qquad \lambda m_\nu \lambda^T = \frac{1}{v^2} M_R^{1/2} R \bar{m}_\nu^2 R^T M_R^{1/2}$$

→ η_{B0} does not depend on U_{MNS} ...

The *CP*-violating phases in the neutrino oscillation and Leptogenesis are independent.

Seesaw Mechanism vs Neutrino oscillation

- The seesaw mechanism is attractive model to explain the observed tiny neutrino mass.
- Without knowing the origin of λ, it is difficult to test the seesaw mechanism from the low energy data.
- Observation of the CP-asymmetry in neutrino oscillations will support Leptogenesis qualitatively, but they are quantitatively independent.

To go one step further?

Top down : Flavor symmetries, Grand Unified Theory...

Instead, we take a bottom up approach as a trial where we reduce the number of the Yukawa couplings as small as possible as long as the experimental results are reproduced (Occam's Razor).

Seesaw Mechanism with Occam's Razor

- ✔ We need only two right-handed neutrinos!

$$\bar{m}_\nu = U_{MNS}^T \lambda^T M_R^{-1} \lambda U_{MNS} v^2$$

(rank[$\bar{m}_\nu$] = min[rank[U_{MNS}], rank[λ], rank[M_R]])

→ the lightest neutrino mass = 0 !

- ✔ Number of real valued parameters

Seesaw Mechanism

M_i	2
$y_{\alpha\beta}$	3
$\lambda_{i\alpha}$	9 = (12-3)

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Low energy theory

M_i	2
$y_{\alpha\beta}$	3
$\bar{m}_{\nu i}$	2
U_{MNS}	5 = 3 + 1 + 1

- ✔ A complex redundant parameter z :

[Normal Hierarchy : $\bar{m}_{\nu 1} = 0$]

$$R = \begin{pmatrix} 0 & \cos z & -\sin z \\ 0 & \sin z & \cos z \end{pmatrix}$$

[Inverted Hierarchy : $\bar{m}_{\nu 3} = 0$]

$$R = \begin{pmatrix} -\sin z & \cos z & 0 \\ \cos z & \sin z & 0 \end{pmatrix}$$

Seesaw Mechanism with Occam's Razor

✓ Minimal Yukawa Structure ? (in diagonalized mass bases)

✗ $\lambda = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$ only one massive neutrino...

✗ $\lambda = \begin{pmatrix} a & a' & 0 \\ b & 0 & 0 \end{pmatrix}$ only one neutrino mixing angle...

✗ $\lambda = \begin{pmatrix} a & a' & 0 \\ b & b' & 0 \end{pmatrix}$ only two neutrino mixing angles...

○ $\lambda = \begin{pmatrix} a & a' & 0 \\ b & 0 & b' \end{pmatrix}$ $\lambda = \begin{pmatrix} a & 0 & 0 \\ b & b' & b'' \end{pmatrix}$ ...

Seesaw Mechanism

M_i	2
$y_{\alpha\beta}$	3
λ_{ia}	5 = (8-3)

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Low energy theory

M_i	2
$y_{\alpha\beta}$	3
$\bar{m}_{\nu i}$	2
U_{MNS}	5 = 3 + 1 + 1

→ we have non-trivial predictions on U_{MNS} and $\bar{m}_{\nu i}$.

['02 Frampton, Glashow, Yangagida, '02 Raidal, Strumia, '04 Ibarra, Ross]

Seesaw Mechanism with Occam's Razor

✔ Do they reproduce the observed 5 parameters ?

✔ Mass differences :

$$\Delta m^2_{21} = 7.59^{+0.20}_{-0.18} \times 10^{-5}\,\mathrm{eV}^2 \ , \qquad \Delta m^2_{31} = 2.45^{+0.09}_{-0.09} \times 10^{-3}\,\mathrm{eV}^2\,(NH) \ ,$$

$$\Delta m^2_{31} = -2.34^{+0120}_{-0.09} \times 10^{-3}\,\mathrm{eV}^2\,(IH) \ ,$$

✔ Mixing Angle :

$$\sin^2\theta_{12} = 0.312^{+0.017}_{-0.015} \ , \qquad \sin^2\theta_{23} = 0.51^{+0.06}_{-0.06}\,(NH) \ , \quad \sin^2\theta_{13} = 0.023^{+0.004}_{-0.004} \ ,$$

$$\sin^2\theta_{23} = 0.52^{+0.06}_{-0.06}\,(IH) \ ,$$

['11 Schwetz, M. Tortola and J. W. F. Valle, '12 Daya Bay]

We put two-zeros in λ

- Redundant parameter "z" is fixed.
- Two relations on U_{MNS} and $\bar{m}_{\nu i}$.

→ 5 (out of 7) parameters remain in U_{MNS} and $\bar{m}_{\nu i}$!

We have sufficient parameters!

Seesaw Mechanism with Occam's Razor

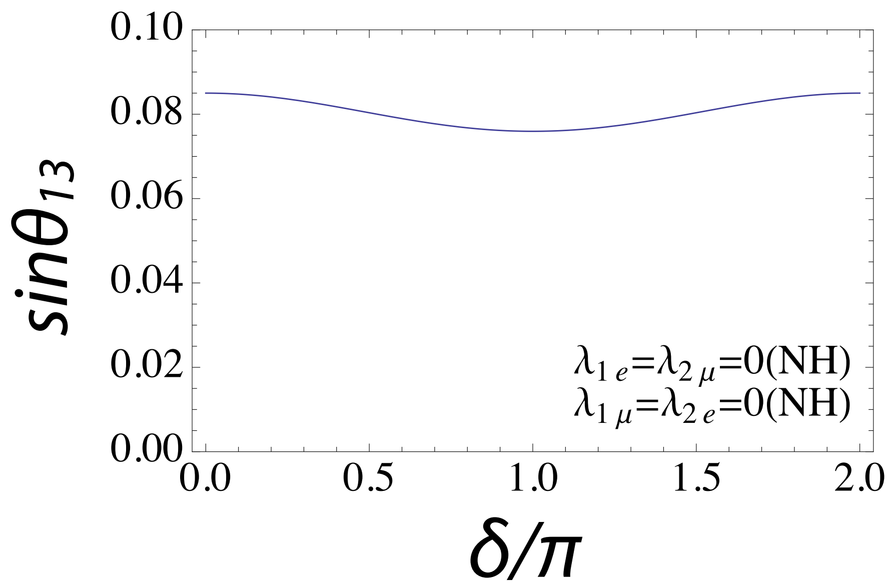
Ex1) $\lambda_{1e} = \lambda_{2\mu} = 0$ or $\lambda_{1\mu} = \lambda_{2e} = 0$ in the normal hierarchy.

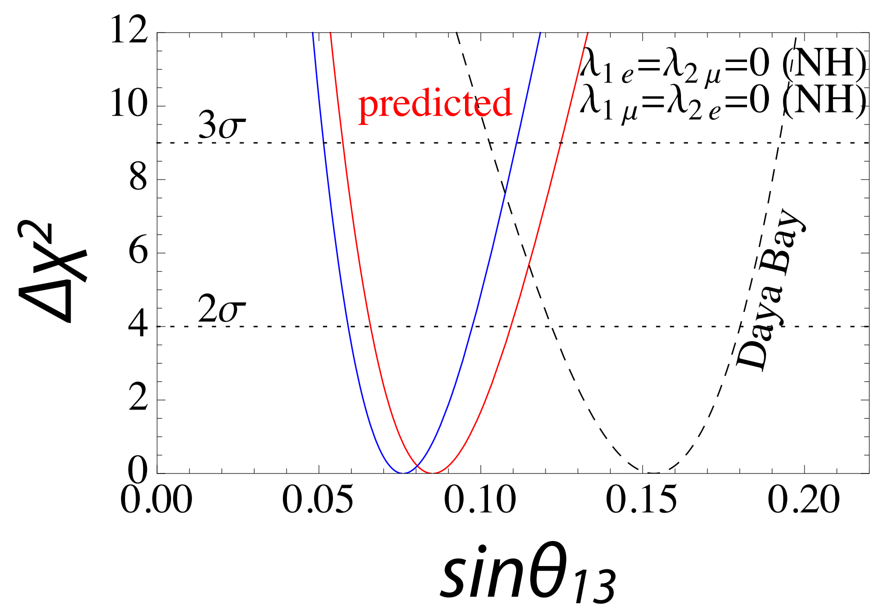
A complex relation on U_{MNS} and $\bar{m}_{\nu i}$.

$$m_3 s_{13} s_{23} e^{-i(\delta+\alpha)} + m_2 s_{12}(c_{12}c_{23} - e^{i\delta} s_{12}s_{13}s_{23}) = 0$$

This condition cannot be satisfied for the observed 5 parameters for any values of α and δ !

For given Δm^2_{21}, Δm^2_{31}, $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$





A bit small $\sin\theta_{13}$ is predicted... → excluded !

Seesaw Mechanism with Occam's Razor

✔ Similarly, all the other possibilities in the normal hierarchy are not consistent with the observed 5 parameters...

For the normal hierarchy with $m_1 = 0$, the Yukawa coupling λ depends on $U_{\alpha 3}$, and two-zero conditions lead to a sharp prediction on $\sin\theta_{13}$, which contradicts with observations.

Explicit Yukawa coupling in the normal hierarchy

$$
\begin{aligned}
\lambda_{1\alpha} &= \frac{1}{v}\sqrt{M}_1(\sqrt{m_2}\,U^*_{\alpha 2}\,c_z - \sqrt{m_3}\,U^*_{\alpha 3}\,s_z)\ , \\
\lambda_{2\alpha} &= \frac{1}{v}\sqrt{M}_2(\sqrt{m_2}\,U^*_{\alpha 2}\,s_z + \sqrt{m_3}\,U^*_{\alpha 3}\,c_z)\ ,
\end{aligned}
$$

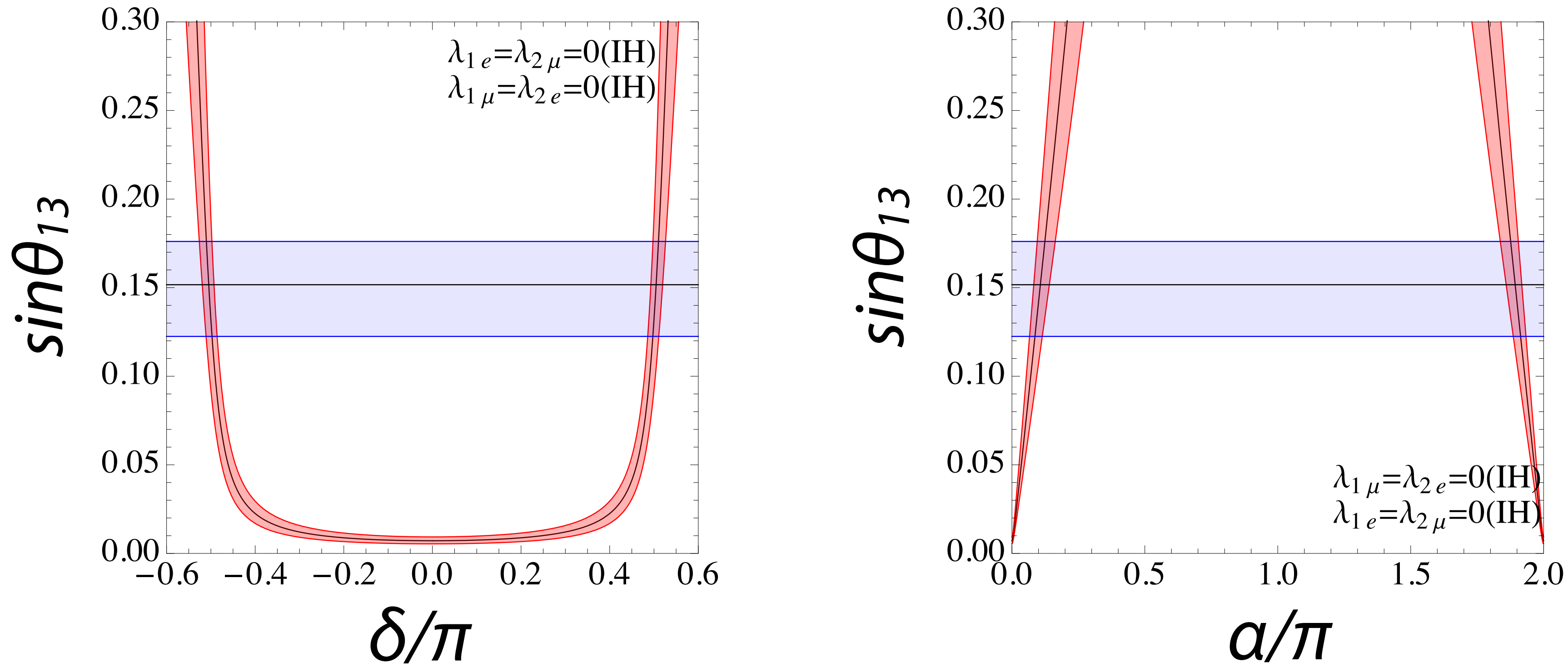
Seesaw Mechanism with Occam's Razor

Ex2) $\lambda_{1e} = \lambda_{2\mu} = 0$ or $\lambda_{1\mu} = \lambda_{2e} = 0$ in the inverted Hierarchy.

A complex relation on U_{MNS} and $\bar{m}_{\nu i}$.

$$m_1 c_{12}(c_{23}s_{12} + c_{12}s_{23}s_{13}e^{i\delta}) - m_2 s_{12}(c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta})e^{i\alpha} = 0 \ .$$

For given Δm^2_{21}, Δm^2_{31}, $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$



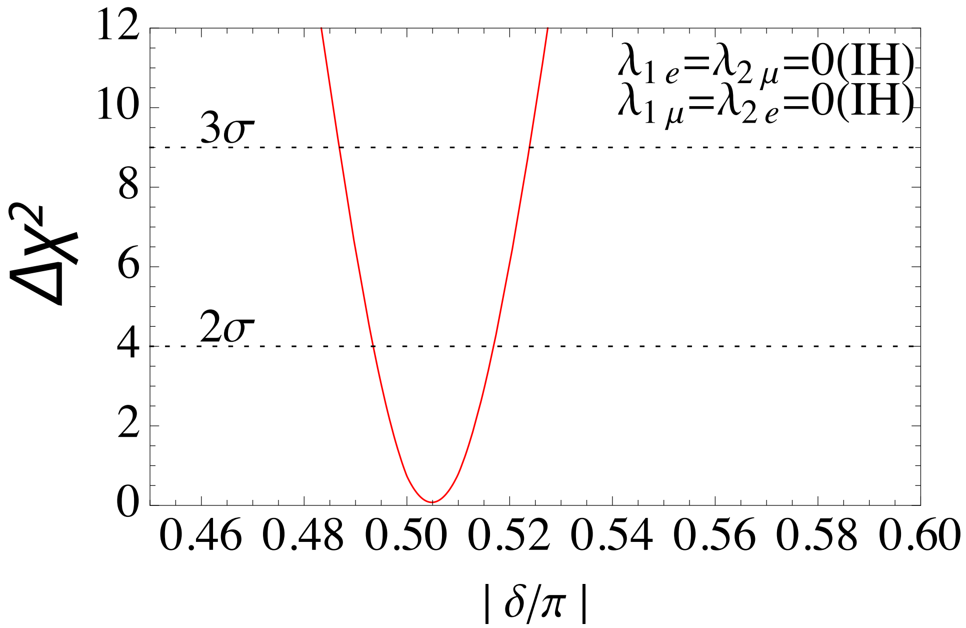
This relation is consistent with data only for $\delta \simeq \pm\pi/2$!

Seesaw Mechanism with Occam's Razor

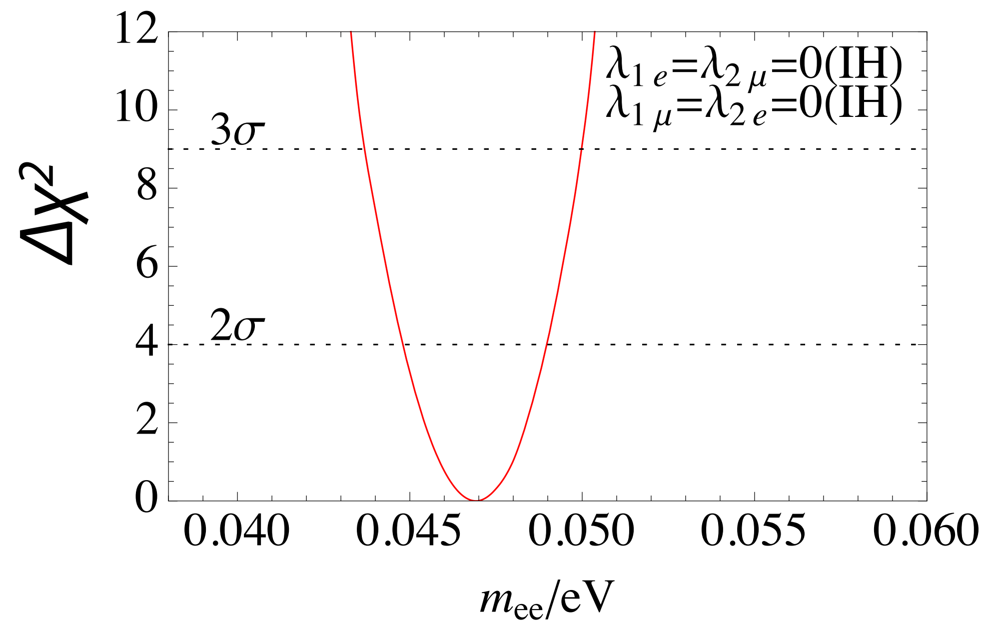
In the inverted hierarchy, we found four consistent possibilities :

$$\lambda_{e2} = \lambda_{\mu 1} = 0 \; (\lambda_{e1} = \lambda_{\mu 2} = 0) \qquad \lambda_{e2} = \lambda_{\tau 1} = 0 \; (\lambda_{e1} = \lambda_{\tau 2} = 0)$$

In these cases, we have very sharp predictions !



$$\delta \simeq \pm\frac{\pi}{2}$$



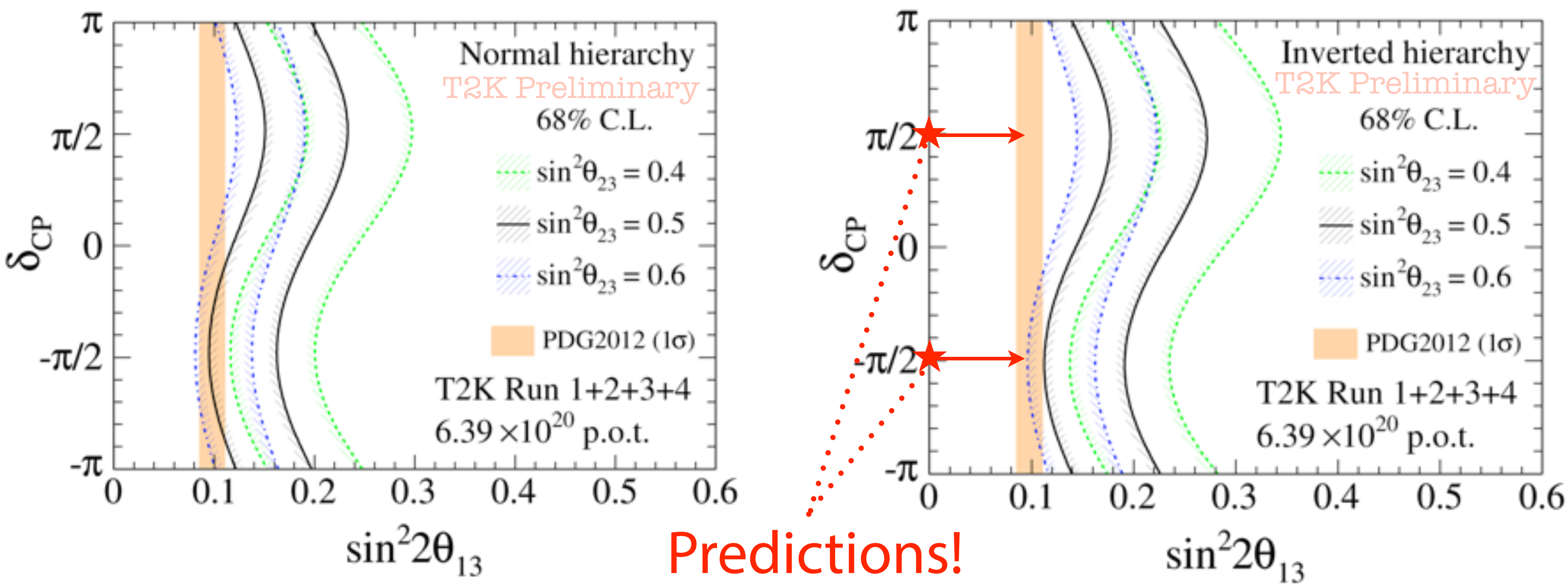
$$m_{ee} \simeq 47\,\mathrm{meV}$$

The effective Majorana neutrino mass

$$m_{ee} \;=\; |m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2|$$

In passing...

Combination of the latest T2K and reactor experiments...



$$\begin{cases} \delta \simeq \pi/2 & \text{is getting excluded...} \\ \delta \simeq -\pi/2 & \text{is getting favored...??} \end{cases}$$

Seesaw Mechanism with Occam's Razor

✓ Implications on Leptogenesis

- Neutrino oscillation : Dirac *CP*-phase δ in U_{MNS}
- Leptogenesis : *CP*-phase of the z in R

They are now interrelated !

$$\eta_{B0} \propto \; m_\nu^{\rm eff} \sin \delta_{\rm eff} = \frac{\Delta m^2_{12}}{\tilde{m}_1} {\rm Im}[c_z^2] \qquad \tilde{m}_1 = (\lambda\lambda^\dagger)_{11} \frac{v^2}{M_R}$$

$${\rm Im}[c_z^2] = \pm s_{12} c_{12} t_{23} s_{13} \sin\delta = \pm \frac{J_{CP}}{c_{13}^2 c_{23}^2}$$

(plus) : $\lambda_{e1} = \lambda_{\mu 2} = 0, \lambda_{e1} = \lambda_{\tau 2} = 0$ (minus) : $\lambda_{e2} = \lambda_{\mu 1} = 0, \lambda_{e2} = \lambda_{\tau 1} = 0$

The observation of the *CP*-violation in the neutrino oscillation directly probe the *CP*-violation in Leptogenesis!

$$\eta_{B_0} \simeq \pm 5.9 \times 10^{-10} \times \left(\frac{M_1}{5 \times 10^{13}\,{\rm GeV}} \right)$$

Summary

- ✔ The seesaw mechanism is an attractive framework which explains the tiny neutrino masses!
- ✔ The seesaw mechanism also makes it possible to explain the Baryon Asymmetry of the universe via Leptogenesis.
- ✔ The seesaw mechanism does not give any particular predictions on the mixing angles and the masses...
- ✔ The *CP*-violation used in Leptogenesis is independent from the *CP*-violation in the neutrino oscillations...

In the spirit of the Occam's Razor, it is possible to reduce the seesaw mechanism down to...

Two right-handed neutrino

Two zeros in the Yukawa coupling λ.

Summary

Once the seesaw mechanism is *shaved* down to this level...

Surprisingly sharp predictions !

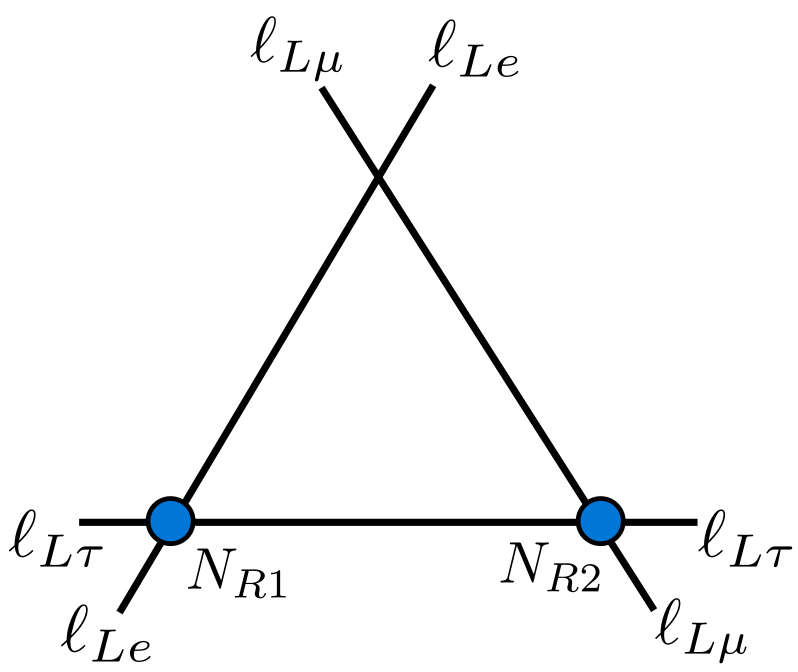
- ✓ One massless neutrino
- ✓ Inverted hierarchy!
- ✓ $\delta \simeq \pm\frac{\pi}{2}$
- ✓ $m_{ee} \simeq 47\,\mathrm{meV}$

The *CP*-phase in the neutrino oscillations directly probes the *CP*-phase in Leptogenesis !

Summary

Any physics behind?

$\lambda_{1\mu} = \lambda_{2e} = 0$



A higher dimensional realization.

The charged leptons are on the branes.

The two right-handed neutrinos reside on the intersections.

The Higgs boson is not localized.

Once the observed δ and m_{ee} are found to be consistent with our predictions, they can be explained by the "surprisingly shaved" seesaw mechanism.

This might reflect the structure of spacetime geometry in higher dimensional theories...

Backup

Sakharov three conditions ('67)

Density operator : $\rho = \Sigma f_n | n >< n |$

$$i \partial\rho/\partial t + [\rho, H] = 0$$

$$\rho(t) = e^{iHt} \rho e^{-iHt}$$

Baryon asymmetry : $<n_B>(t) = Tr[\rho(t) B]$ with $<n_B>(0) = 0$

Sakharov three conditions ('67)

For $[H, B] = 0$: $<n_B>(t) = <n_B>(0) = 0$ Sakharov #1

For $[H, C] = 0$: $<n_B>(t) = - <n_B>(t) \rightarrow <n_B>(t) = 0$

Sakharov #2

For $[H, CP] = 0$: $<n_B>(t) = - <n_B>(t) \rightarrow <n_B>(t) = 0$

In thermal equilibrium : Baryon production rate

= Inverse Baryon production rate

Sakharov #3

Generic two-zero conditions

Normal Hierarchy

$$\lambda_{1\alpha} = 0$$

$$\tan z = \frac{\sqrt{m_2}\, U^*_{\alpha 2}}{\sqrt{m_3}\, U^*_{\alpha 3}} \ ,$$

$$\lambda_{2\alpha} = 0$$

$$\tan z = -\frac{\sqrt{m_3}\, U^*_{\alpha 3}}{\sqrt{m_2}\, U^*_{\alpha 2}}$$

$$\lambda = \begin{pmatrix} a & a' & 0 \\ b & 0 & b' \end{pmatrix}$$

$$\rightarrow \quad m_2\, U_{\alpha 2}\, U_{\alpha' 2} + m_3\, U_{\alpha 3}\, U_{\alpha' 3} = 0$$

$$\lambda = \begin{pmatrix} a & 0 & 0 \\ b & b' & b'' \end{pmatrix}$$

$$\rightarrow \quad U_{\alpha 2}\, U_{\alpha' 3} = U_{\alpha 3}\, U_{\alpha' 2}$$

Inverted Hierarchy

$$\lambda_{1\alpha} = 0$$

$$\tan z = \frac{\sqrt{m_2}\, U^*_{\alpha 2}}{\sqrt{m_1}\, U^*_{\alpha 1}} \ ,$$

$$\lambda_{2\alpha} = 0$$

$$\tan z = -\frac{\sqrt{m_1}\, U^*_{\alpha 1}}{\sqrt{m_2}\, U^*_{\alpha 2}}$$

$$\lambda = \begin{pmatrix} a & a' & 0 \\ b & 0 & b' \end{pmatrix}$$

$$\rightarrow \quad m_2\, U_{\alpha 2}\, U_{\alpha' 2} + m_1\, U_{\alpha 1}\, U_{\alpha' 1} = 0$$

$$\lambda = \begin{pmatrix} a & 0 & 0 \\ b & b' & b'' \end{pmatrix}$$

$$\rightarrow \quad U_{\alpha 2}\, U_{\alpha' 1} = U_{\alpha 1}\, U_{\alpha' 2}$$

Generic two-zero conditions

Definitions of the U_{MNS}

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \times \mathrm{diag}(1,\ e^{i\frac{\alpha_{21}}{2}},\ \cancel{e^{i\frac{\alpha_{31}}{2}}})\ .$$

$$\frac{|U_{e2}|^2}{|U_{e1}|^2} \equiv \tan^2\theta_{12}; \qquad \frac{|U_{\mu 3}|^2}{|U_{\tau 3}|^2} \equiv \tan^2\theta_{23}; \qquad U_{e3} \equiv \sin\theta_{13}e^{-i\delta},$$

Allowed Yukawa couplings

In the inverted hierarchy, we found four consistent possibilities :

$\lambda_{e2} = \lambda_{\mu 1} = 0$ ($\lambda_{e1} = \lambda_{\mu 2} = 0$)

$$\lambda = \begin{pmatrix} 0.12 \times e^{-0.053\,i} & 0 & 0.028 \times e^{1.5\,i} \\ 0 & 0.28 \times e^{3.0\,i} & 0.29 \times e^{-0.12\,i} \end{pmatrix} \begin{matrix} \text{x} \; (M_1/10^{13}\,\text{GeV})^{1/2} \\ \text{x} \; (M_2/10^{14}\,\text{GeV})^{1/2} \end{matrix} \updownarrow$$

$$z = 0.98 \times e^{-3.1\,i}$$

$\lambda_{e2} = \lambda_{\tau 1} = 0$ ($\lambda_{e1} = \lambda_{\tau 2} = 0$)

$$\lambda = \begin{pmatrix} 0.12 \times e^{-0.049\,i} & 0.027 \times e^{-1.6\,i} & 0 \\ 0 & 0.28 \times e^{3.0\,i} & 0.29 \times e^{-0.11\,i} \end{pmatrix} \begin{matrix} \text{x} \; (M_1/10^{13}\,\text{GeV})^{1/2} \\ \text{x} \; (M_2/10^{14}\,\text{GeV})^{1/2} \end{matrix} \updownarrow$$

$$z = 0.98 \times e^{-3.1\,i}$$

In these cases, we have non-trivial very sharp predictions

$$\delta \simeq \pm\pi/2 \qquad m_{ee} \simeq 47\,\text{meV}$$

Putting zero ?

In the quark sector, the Cabbibo angle is a parameter.

The Cabbibo angle *can be derived* if we put zero in M_d !

$$M_u = \begin{pmatrix} m_u & 0 \\ 0 & m_c \end{pmatrix} \qquad M_d = \begin{pmatrix} 0 & \sqrt{m_d m_s} \\ \sqrt{m_d m_s} & m_s \end{pmatrix}$$

→ $\sin\theta_C = (m_d/m_s)^{1/2} \sim 0.22$!

[S. Weinberg, HUTP-77-A057, Trans.New York Acad.Sci.38:185-201, 1977]

Leptogenesis

$$\epsilon = \frac{\Gamma[N \to \ell + h] - \Gamma[N \to \ell^\dagger + h^\dagger]}{\Gamma[N \to \ell + h] + \Gamma[N \to \ell^\dagger + h^\dagger]}$$

$$\simeq \frac{3}{16\pi} \frac{M_1}{v^2} \frac{\mathrm{Im}[(\lambda m_\nu \lambda^T)_{11}]}{(\lambda\lambda^\dagger)_{11}}$$

$$\frac{n_B}{n_\gamma} = \frac{28}{79} \frac{n_{B-L}}{n_\gamma} = \frac{28}{79} \frac{n_L}{n_\gamma}\bigg|_{N_R \mathrm{decay}}$$

ν_e appearance

$$P(\nu_\mu \rightarrow \nu_e) \cong \sin^2 2\theta_{13}\, T_1 - \alpha \sin 2\theta_{13}\, T_2 + \alpha \sin 2\theta_{13}\, T_3 + \alpha^2\, T_4$$

$$T_1 = \sin^2\theta_{23} \sin^2[(1-x_\nu)\Delta]/(1-x_\nu)^2$$

$$T_2 = \sin\delta \sin 2\theta_{12} \sin 2\theta_{23} \sin\Delta \sin(x_\nu\Delta)/x_\nu \sin[(1-x_\nu)\Delta]/(1-x_\nu)$$

$$T_3 = \cos\delta \sin 2\theta_{12} \sin 2\theta_{23} \cos\Delta \sin(x_\nu\Delta)/x_\nu \sin[(1-x_\nu)\Delta]/(1-x_\nu)$$

$$T_4 = \cos^2\theta_{23} \sin^2 2\theta_{12} \sin^2(x_\nu\Delta)/x_\nu^2$$

$$\Delta \equiv \Delta m^2_{31} L/4E,\ \alpha \equiv \Delta m^2_{21}/\Delta m^2_{31} \sim 1/30,\ x_\nu \equiv 2\sqrt{2} G_F N_e E/\Delta m^2_{31}$$