格子QCD講義

橋本省二 (KEK, 総研大)

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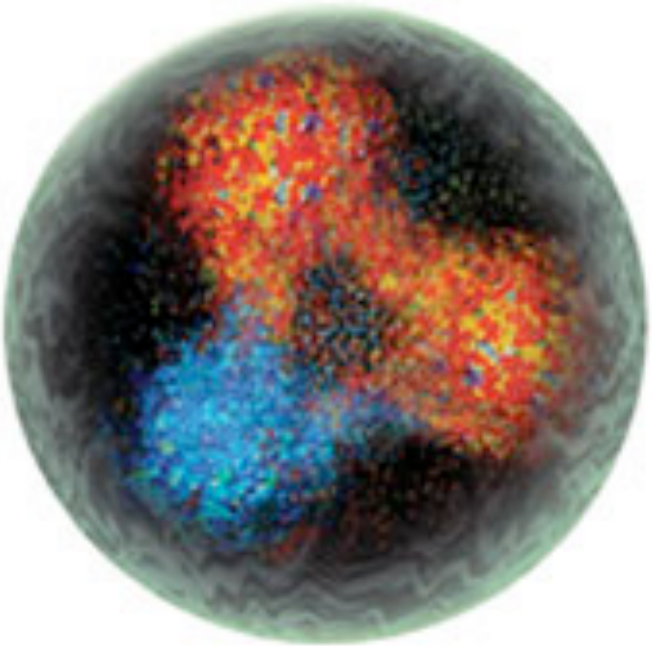
計算基礎科学連携拠点
Joint Institute for
JICFuS Computational Fundamental Science

国立大学法人
総合研究大学院大学
The Graduate University for Advanced Studies [SOKENDAI]

理解したいこと

核子はこんな感じ？



３つのクォークの束縛状態…, 量子色力学 (QCD)。
実態はもっとややこしい。

もくじ

目標：素粒子現象におけるハドロン不定性と格子ゲージ理論による計算手法、その誤差について理解する。

0. 歴史

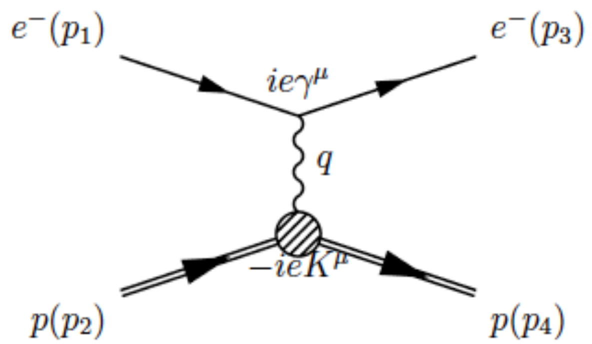
1. 量子色力学(QCD)の性質
   - 摂動法、繰り込み群、クォーク閉じ込め、自発的対称性の破れ

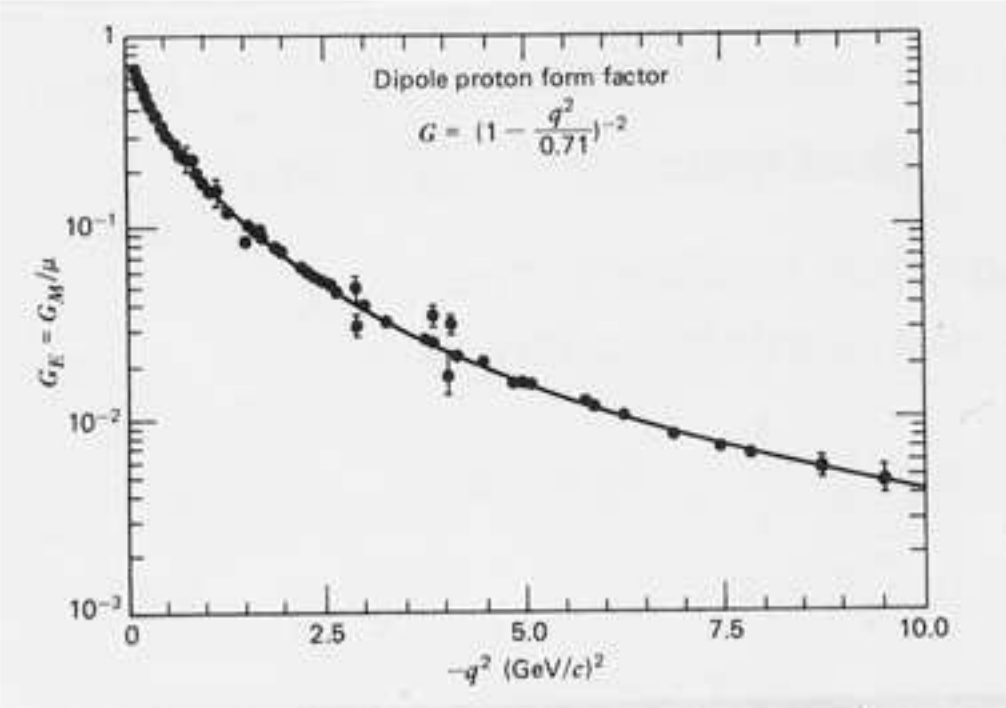
2. 格子ゲージ理論の基礎
   - 定式化、数値計算法、誤差の要因

3. 素粒子現象論への応用
   - ハドロン不定性とは、π, K 中間子の物理、核子の性質

0. A bit of history

ep scattering = proton seems to have internal structure





A proton extends as exp(- r/r_0); r_0 ~ 1 fm

Form factor

$$\tau = \frac{Q^2}{4M^2} \quad \epsilon = [1 + 2(1+\tau)\tan^2\frac{\theta_e}{2}]^{-1}$$

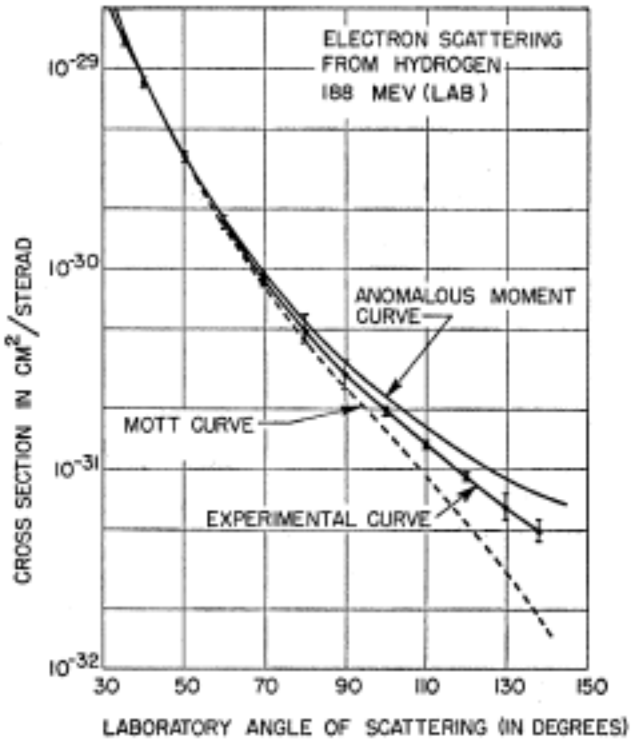
$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_M \times \left[G_E^2 + \frac{\tau}{\epsilon}G_M^2\right]\frac{1}{(1+\tau)}$$

Mott scattering (point particle)

Form factors
(representing the internal structure)

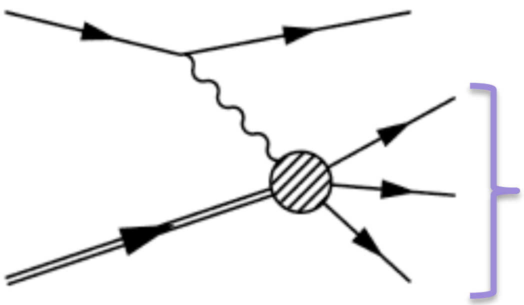
$$\sigma(\theta_e) = \sigma_M \left|\int \rho(\vec{r})e^{i\vec{q}.\vec{r}}d^3\vec{r}\right|^2 = \sigma_M |F(\mathrm{q})|^2$$

$$\text{dipole}: \; F(q^2) = \frac{1}{(1-q^2/q_0^2)^2} \longleftrightarrow \rho(r) \sim \exp(-r/r_0)$$

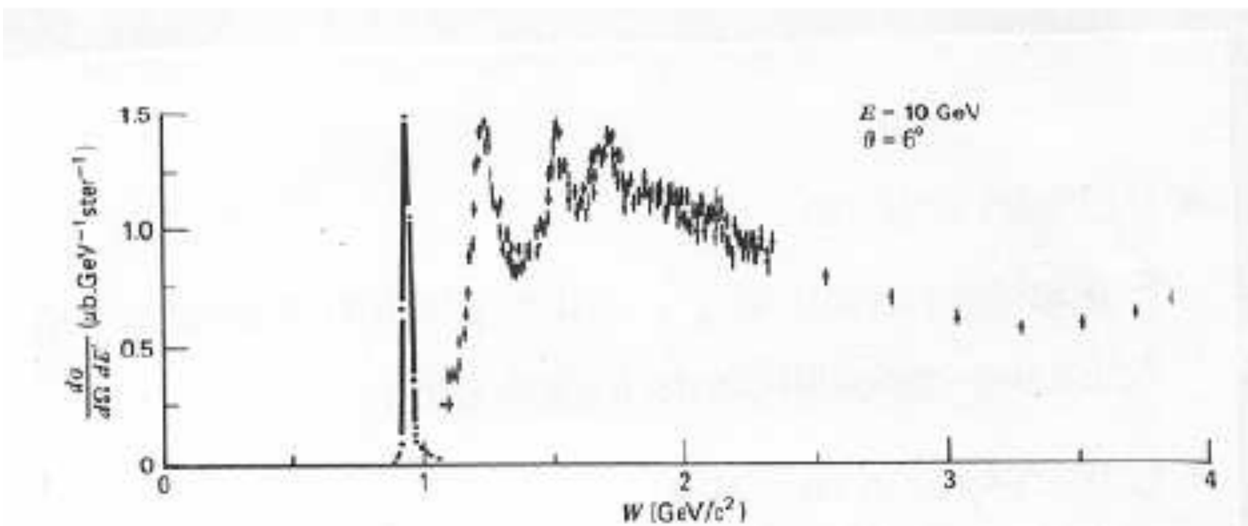


Hofstadter (1955)

Generates resonances at higher energies: ep → eX

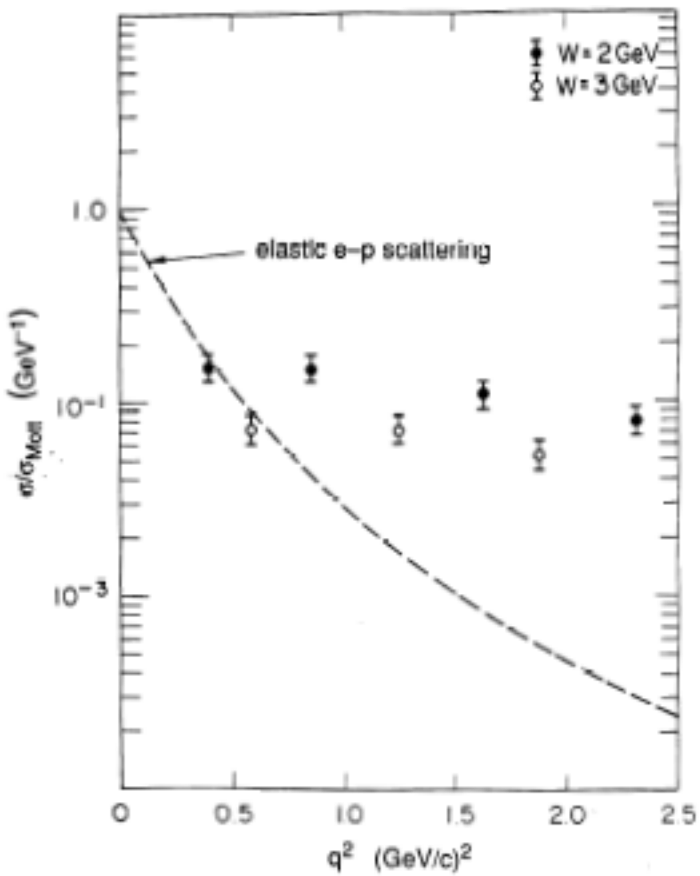


W: invariant mass of produced particles



Again, suggests some internal structures

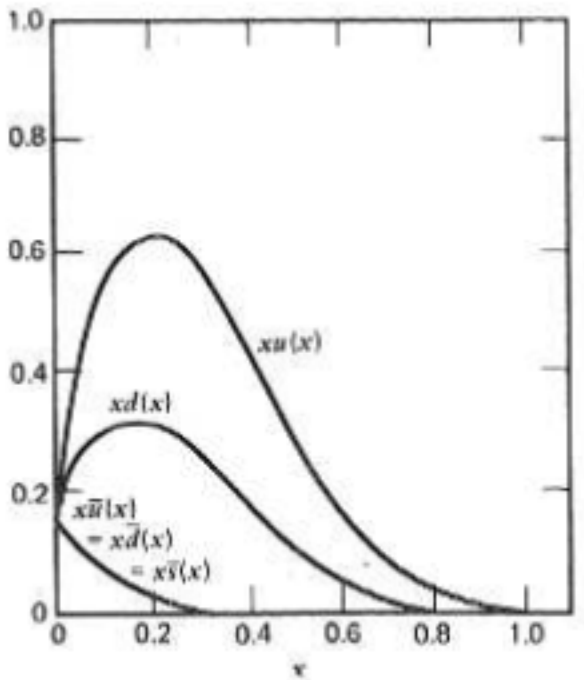
At higher energies, electron looks like hitting a free "parton" inside proton.



Deep Inelastic Scattering (DIS)
Friedman, Kendall, Taylor (1969)

$$\frac{d^2\sigma}{d\Omega dE'} = \left.\frac{d^2\sigma}{d\Omega dE'}\right|_{\mathrm{point}} \underbrace{\left(2F_1 \tan^2\frac{\theta}{2} + F_2\right)}_{\text{structure functions}}$$

At higher energies, electron looks like hitting a free "parton" inside proton.



$$F_2 = x \sum_q e_q^2\, q(x)$$

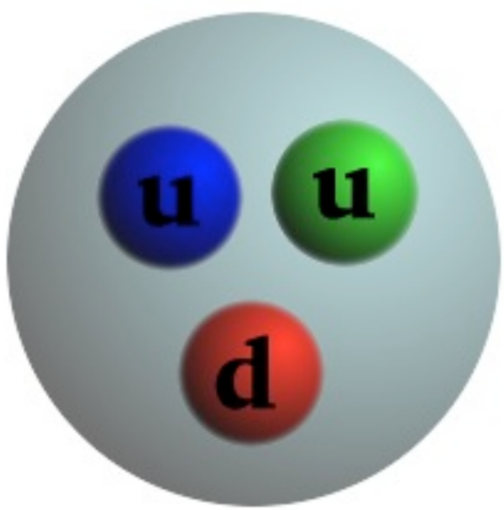
parton distribution function (PDF) =

probability to find a quark q in nucleon with momentum fraction x.

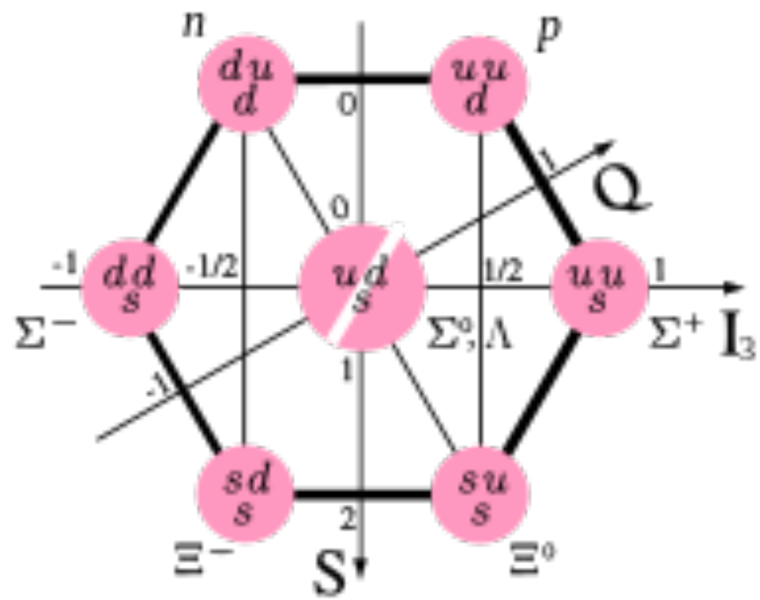
Each parton carries *roughly* 1/3 of proton's momentum.

<u>Quark model</u> Gell-Mann, Zweig (1964)

Okay. There are three quarks inside. up or down

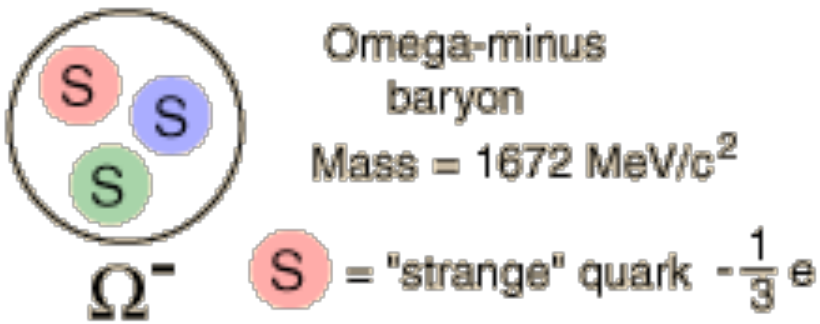


Can explain the baryon spectrum.



Quark model

Need to have three internal degrees of freedom
= color



Otherwise, forbidden by Pauli's exclusion principle.

Quarks

Requirements:

1. have fractional charge +2/3 e, -1/3 e
2. have internal degrees of freedom (=3)
3. may not appear as an isolated particle
4. (at high energy) behave as a free particle inside a proton

Model (or dynamics) to fulfill all of these ➔ QCD

1. Properties of Quantum Chromodynamics (QCD)

Perturbation theory, Renormalization group,
Quark confinement, Spontaneous symmetry breaking

Dirac equation

QED
(for electron)

$$\left(\gamma^{\mu}\left(i\hbar\partial_{\mu}-\frac{e}{c}A_{\mu}\right)-mc\right)\psi=0$$

QCD
(for quark)

$$\left(\gamma^{\mu}\left(i\partial_{\mu}-gA_{\mu}\right)-mc\right)\psi=0$$

3x3 matrix

three degrees of freedom

Maxwell's equation

QED
(for photon)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0$$

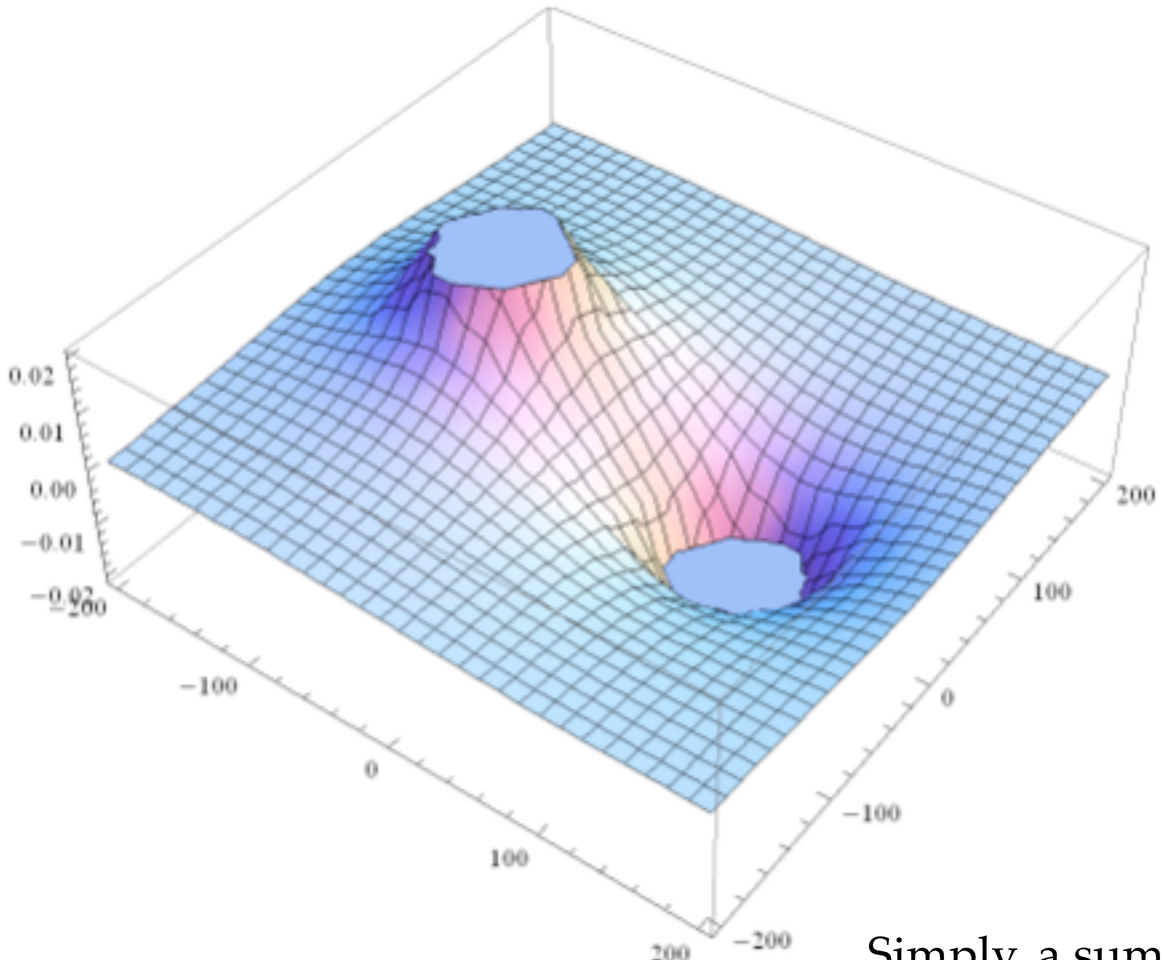
QCD
(for gluon)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$(\partial_\mu + igA_\mu)F^{\mu\nu} = j^\nu, \quad \varepsilon^{\mu\nu\rho\sigma}(\partial_\mu + igA_\mu)F_{\nu\rho} = 0$$

gauge field itself plays the role of a source
= non-linear equation

Coulomb potential



Simply, a sum of two sources = linear
Not the case for QCD!

Perturbation theory

- Non-linear system cannot be solved analytically (in general).
- Use the perturbation theory. What is it?
  - In the language of canonical quantization (= second quantization), only the free field can be solved easily. Equivalent to the harmonic oscillator:

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \quad = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

  - Other operators are treated as a perturbation. The eigenvalues and wave functions are expanded in powers of λ.

$$\delta H = \lambda\hat{x}^4$$

Perturbation theory

- Non-linear system cannot be solved analytically (in general).
- Use the perturbation theory. What is it?
  - In the language of path-integral quantization, only the Gaussian integral can be calculated analytically. Others are estimated by an expansion.

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2-\lambda x^4} = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2}\left(1-\lambda x^4+\cdots\right)$$

  - Can be reduced to the Gaussian integral.

Quantum "fields"

1, 2, 3 for Quantum Field Theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2$$

1. Reinterpret "ϕ" as "x" in Quantum Mechanics
   - Commutation relation
2. Fourier transform
   - The system becomes a set of independent harmonic oscillator for each momentum mode.
3. Solve the harmonic oscillator
   - Eigenstates |0>, |1>, |2>, … for each momentum mode = the number of "particles"

Perturbation theory

- Based on the states identified in the free theory, calculate

$$\langle p_1', p_2', \ldots | \mathcal{L}_{int} | p_1, p_2, \ldots \rangle$$

$\lambda\phi^4$

1 "particle" in the momentum mode p_1

- Feynman rules: easy book-keeping device

QCD perturbation theory

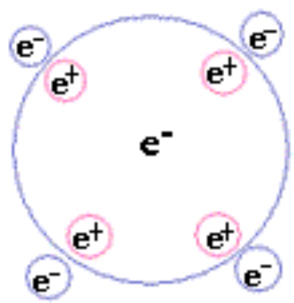
- A lot more complicated, due to…
  - Extra gauge degrees of freedom. Gauge fixing is necessary.
  - Fadeev-Popov ghosts for non-Abelian gauge fields.

- Divergences appear. Need “renormalization”. Before doing that, need “regularization”.
- (perturbative) “renormalizability”

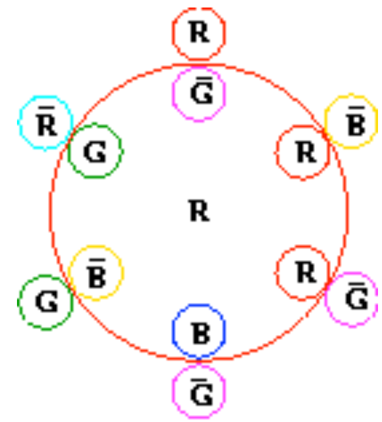
- We don’t want to go through these. Forget about everything, and jump to the consequences.

Anti-screening

Try to measure the coupling constant…

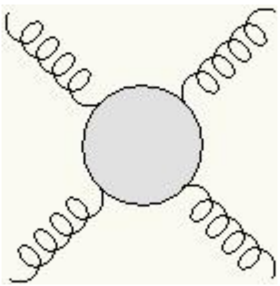


Vacuum polarization weakens the EM charge at long distances



Self-interaction enhances the color charge

Renormalization group



- Scattering amplitude
  - A function of external momenta, coupling constants and a cutoff.

$$A(s,t,u;g_0^2,\Lambda)$$

  - Require that the scattering amplitude does not depend on Λ. Tune the coupling constants.

$$A(s,t,u;g_0^2,\Lambda) = A(s,t,u;g'^2_0,\Lambda')$$

  - Coupling constant is determined as a function of Λ. Input the experimental number at one point of Λ.

$g_0^2(\Lambda)$: "running coupling constant"

Renormalization group

- Two interpretations
  - $g^2(\Lambda)$: bare coupling is determined as a function of the cutoff.
  - $g^2(\mu)$: renormalized coupling is determined depending on the scale of the physical process.

$$A(s,t,u;g_0^2,\Lambda)\big|_{s=t=u=\mu^2} = A_0$$

(tree level amplitude on the RHS) and remove Λ in favor of g_0^2.

$$g^2\left(1+cg^2\ln\frac{\Lambda^2}{q^2}\right)\Bigg|_{q^2=\mu^2} = g^2(\mu)$$

$$g^2\left(1+cg^2\ln\frac{\Lambda^2}{q^2}\right) = g^2\left(1+cg^2\ln\frac{\Lambda^2}{\mu^2}\right)\left(1+cg^2\ln\frac{\mu^2}{q^2}\right) = \overbrace{g^2(\mu)\left(1+cg^2\ln\frac{\mu^2}{q^2}\right)}^{\text{Must be independent of }\mu}$$

$$g^2\left(1+cg^2\ln\frac{\Lambda^2}{q^2}\right)\Bigg|_{q^2=\mu^2} = g^2(\mu)$$

$$g^2\left(1+cg^2\ln\frac{\Lambda^2}{q^2}\right) = g^2\left(1+cg^2\ln\frac{\Lambda^2}{\mu^2}\right)\left(1+cg^2\ln\frac{\mu^2}{q^2}\right) = \overbrace{g^2(\mu)\left(1+cg^2\ln\frac{\mu^2}{q^2}\right)}^{\text{Must be independent of }\mu}$$

But, the whole thing depends on q.
= running coupling

The term like $\ln(q^2/\mu^2)$ vanishes when $q^2=\mu^2$, and (one may hope that) the perturbative expansion converges better. Better to choose μ close to the external momenta rather than taking arbitrarily.

Renormalization group

- μ-dependence of the coupling constant

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln[\ln(\mu^2/\Lambda^2)]}{\ln(\mu^2/\Lambda^2)} + \ldots\right]$$

  - Λ_{QCD} is called the QCD scale. It depends on the renormalization scheme (the way to remove Λ).

- obtained from the Renormalization Group Equation

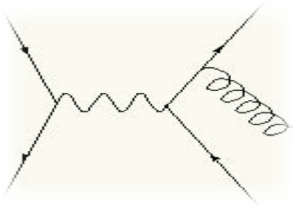
$$\left(\mu\frac{\partial}{\partial\mu} + \mu\frac{d\alpha_s}{d\mu}\frac{\partial}{\partial\alpha_s}\right) R(\mu,\alpha_s) = 0$$

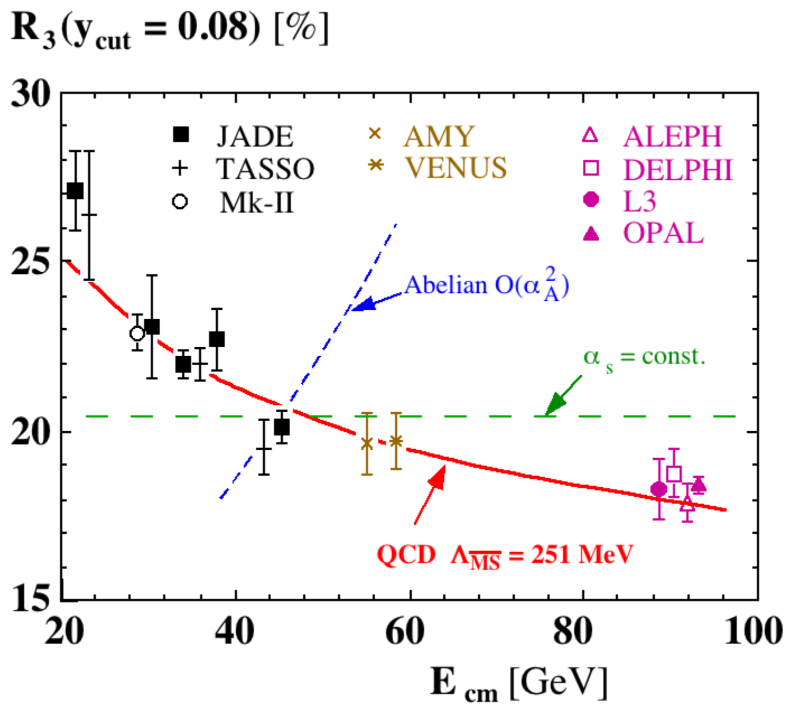
$$\beta(\alpha_s) \equiv -\mu\frac{d\alpha_s}{d\mu} = \beta_0\alpha_s^2 + \beta_1\alpha_s^3 + \cdots$$

Running coupling

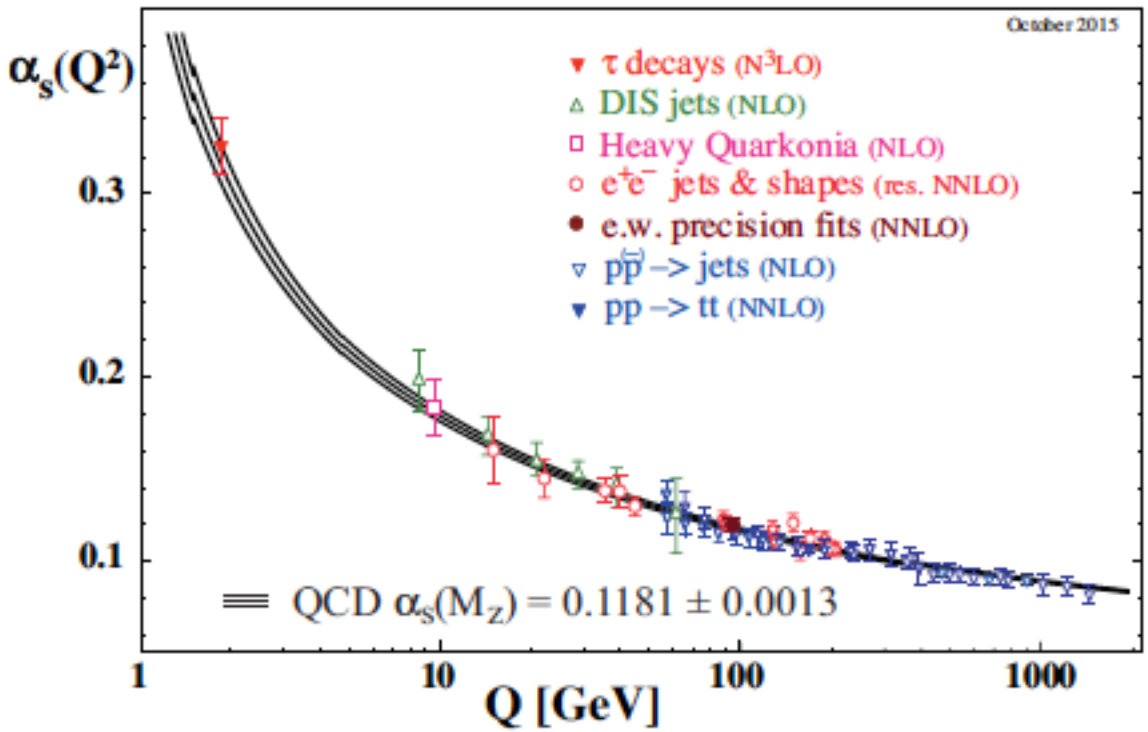
- Confirmed in the physical processes.

$$R_3 = \frac{\sigma\left(e^+e^- \to 3\text{-jets}\right)}{\sigma\left(e^+e^- \to \text{hadrons}\right)} = C_1\alpha_s(\mu^2) + \ldots$$





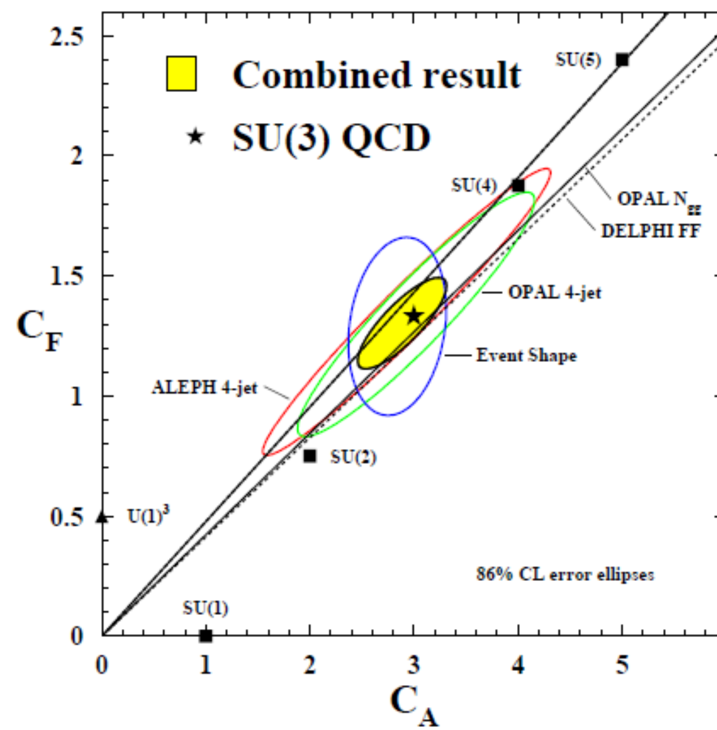
Running coupling



More tests of QCD

- Including 4 jets
  - Sensitivity to the 3-gluon vertex
  - Can test the group structure: SU(3) or not

$$C_A = N = 3,\ C_F = \frac{N^2-1}{2N} = \frac{4}{3}$$

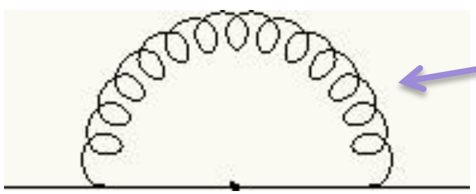


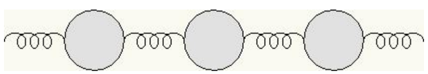
Plots from Bethke, Prog Part Nucl Phys 58 (2007) 351.

Going to low energies

Coupling constant grows. Is that the only problem?

- Perturbative expansion fails to converge. No such thing as the quark pole mass.





$$\sim \int \frac{dk^2}{k^2} f(k^2)[\beta_0 \alpha_s(\mu) \ln(k^2/\mu^2)]^n$$

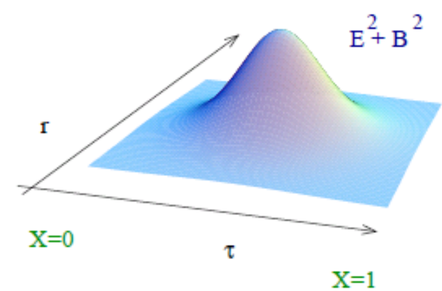
$$\sim (\beta_0 \alpha_s)^n n!$$

  - Higher order terms are increasingly more important.

Going to low energies

Coupling constant grows. Is that the only problem?

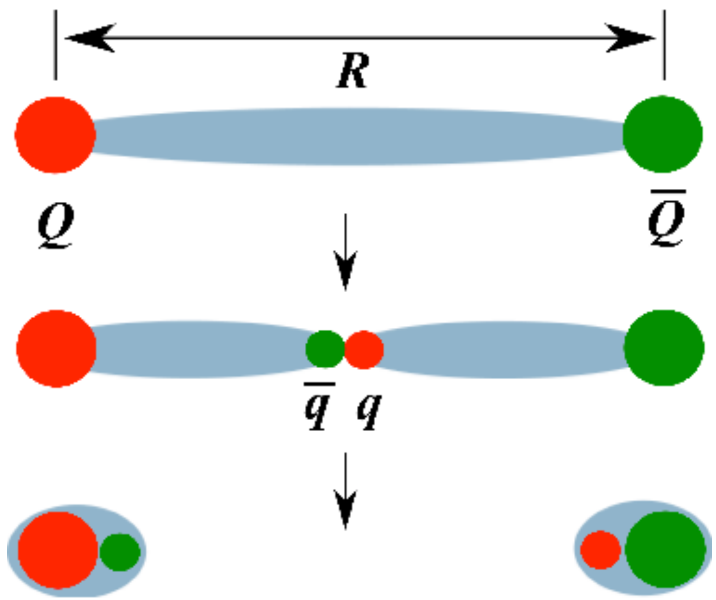
- Non-perturbative configurations, such as instantons = topological excitation



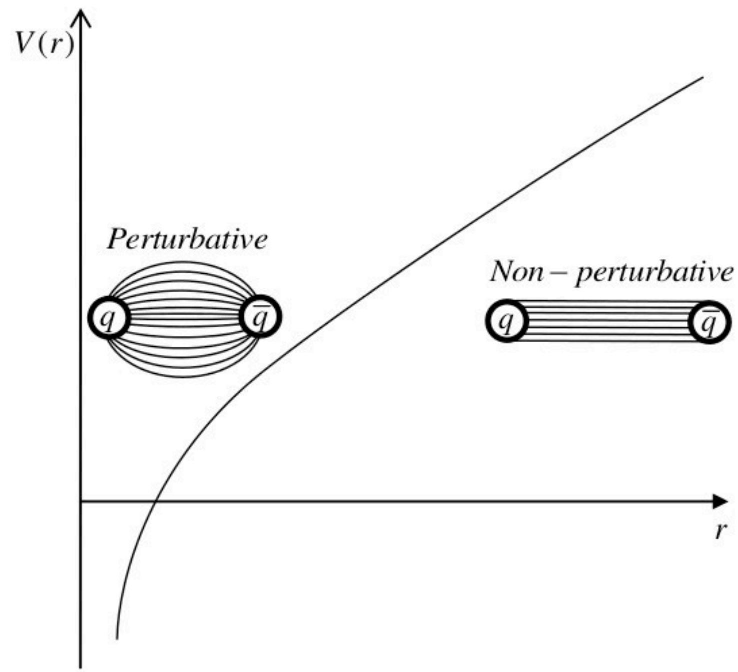
  - Cannot be written by a superposition of plane-waves.
  - Associate fermion zero-modes are essential for chiral symmetry breaking.

Quark confinement

Isolated quarks can never be observed.



Quark confinement



Linear
= flux tube

Coulomb

More details after the introduction of lattice.

Chiral symmetry breaking

- Chiral symmetry
  - Symmetry under $\delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5, \delta\psi = i\alpha\gamma_5\psi$
  - Massless Lagrangian is invariant

$$S = \int d^4x \left[\bar{\psi}(x)\gamma_\mu\partial_\mu\psi(x) + \cancel{m\bar{\psi}(x)\psi(x)} \right]$$

  - Fermion field can be decomposed into R and L

$$\psi_R = \frac{1+\gamma_5}{2}\psi, \psi_L = \frac{1-\gamma_5}{2}\psi$$

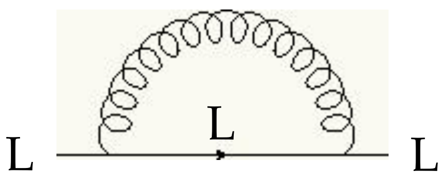
  - chiral rotation is $\delta\psi_R = i\alpha\psi_R, \delta\psi_L = -i\alpha\psi_L$

Chiral symmetry breaking

- Gauge interaction preserves chiral symmetry.

$$\boxed{\bar{\psi}_R \gamma_\mu D_\mu \psi_R + \bar{\psi}_L \gamma_\mu D_\mu \psi_L}$$

  - No right-handed quarks can change to left-handed by emitting a gluon.



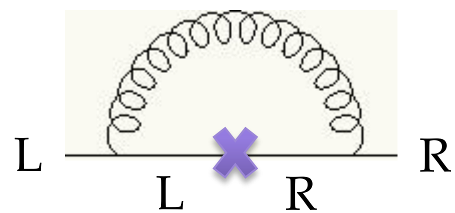
- Mass term breaks chiral symmetry.

$$\boxed{m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)}$$

  - Chiral symmetry breaking = mass generation.

Chiral symmetry breaking

- Then, how can the mass be generated?
  - triggered by a small mass term?



  - Or, spontaneous… vacuum expectation value

$$\langle \bar{\psi}\psi \rangle \neq 0$$

due to non-perturbative effect. There is a class of background gauge field (instantons) that connects L and R.

Some details after the introduction of lattice.

2. Lattice gauge theory

2.1 The basics

lattice, gauge symmetry, inputs

Goal

- QCD becomes non-perturbative at low energies. Perturbation theory cannot reveal the important part of the hadronic phenomena.
  - hadron masses, interactions, …

- Try to construct a framework that enables fully non-perturbative calculation.
  - One may introduce numerical methods.
  - No obvious way to introduce the momentum cutoff that fully respects gauge invariance.
  - Go back to the coordinate space = Lattice gauge theory.

Wilson (1974)

QCD Lagrangian

- SU(3) gauge theory
- plus, quarks (up, down, strange, …)

$$S = \int d^4x \left\{ \frac{1}{4} \mathrm{Tr}\, F_{\mu\nu}^2 + \sum_f \bar{\psi}_f (\not{D} + m_f) \psi_f \right\},$$

$$Z = \int [dA_\mu] \prod_f [d\psi][d\bar{\psi}] \exp\left[-S\right]$$

- Field strength $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$

$$D_\mu = \partial_\mu - igT^a A_\mu^a$$

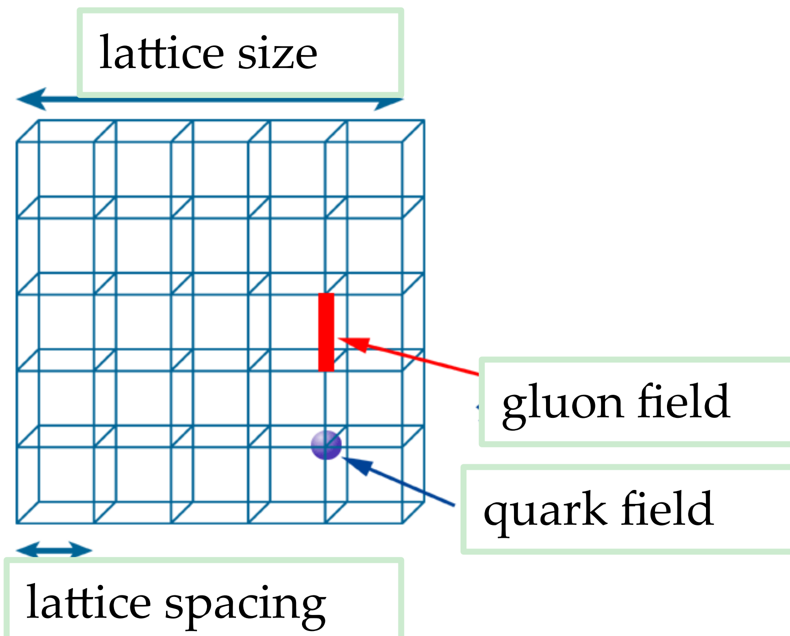
Non-Abelian nature

- Redefine on a 4D lattice

The lattice

<u>4D Lattice</u>

- of size $(L/a)^3$x(T/a), typically 32^3x64 or 64^3x128.
- lattice spacing determined later.



$$A_\mu(x) \to U_\mu(x)$$

$$\psi(x)$$

Gauge invariance

Gauge symmetry

- invariance under local SU(3) transformation
- guaranteed by introducing "link variables" (gauge field)

$$\bar{\psi}(x) \qquad\qquad\qquad\qquad \psi(x+\hat{\mu})$$

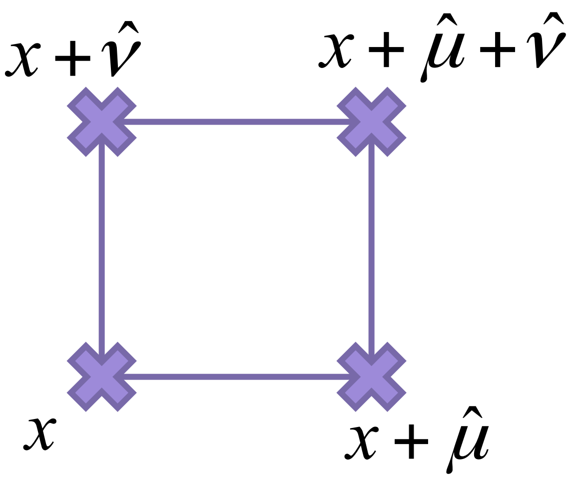
$$\bar{\psi}(x)V^{\dagger}(x) \quad V(x)U_{\mu}(x)V^{\dagger}(x+\hat{\mu}) \quad V(x+\hat{\mu})\psi(x+\hat{\mu})$$

$$U_{\mu}(x) = \exp\left[igaA_{\mu}(x)\right] = 1 + igaA_{\mu}(x) + \ldots$$

$$A_{\mu}(x) \rightarrow V(x)\left[A_{\mu}(x) + \frac{i}{g}\partial_{\mu}\right]V^{\dagger}(x)$$

Gauge field

- Built in the gauge link $\boxed{U_\mu(x) = \exp\left[igaA_\mu(x)\right] = 1 + igaA_\mu(x) + \ldots}$
  - SU(3) matrices
  - Gauge invariance guaranteed by connecting them.



$$\begin{aligned}
&\mathrm{Tr}\left[U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)\right] \\
&\approx \mathrm{Tr}\left[e^{igaA_\mu} e^{iga(A_\nu + a\partial_\mu A_\nu)} e^{-iga(A_\mu + a\partial_\nu A_\mu)} e^{-igaA_\nu}\right] \\
&\approx \mathrm{Tr}\left[e^{iga^2(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2a^2[A_\mu, A_\nu]}\right] = \mathrm{Tr}\left[e^{iga^2 F_{\mu\nu}}\right] \\
&= \mathrm{Tr}[1] - \frac{1}{2} g^2 a^4 \mathrm{Tr}\left[F_{\mu\nu}^2\right] + \ldots
\end{aligned}$$

Gauge action

Should go back to the continuum, by taking $a \to 0$

$$S = \frac{6}{g^2} \sum_x \sum_{\mu<\nu} \left[1 - \frac{1}{3} \mathrm{Re}\,\mathrm{Tr} \left[U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x) \right] \right]$$

$$\to a^4 \sum_x \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{Tr} \left[F_{\mu\nu}^2 \right]$$

$$= \int d^4x \, \frac{1}{4} \left(F_{\mu\nu}^a \right)^2$$

- Coupling constant $\beta = 6/g^2$
  - Corresponds to 1/kT in the statistical model.

Partition function

- Integrate over SU(3) variables U, rather than A

$$Z = \int [dU_\mu] \prod_f [d\psi][d\bar{\psi}] \exp\left[-S_g - \int d^4x \sum_f \bar{\psi}_f (D[U] + m_f) \psi_f \right]$$
$$= \int [dU_\mu] \prod_f \det(D[U] + m_f) \exp[-S_g]$$

- Fermion fields are anti-commuting, giving the determinant when integrated out

Heavy quark potential

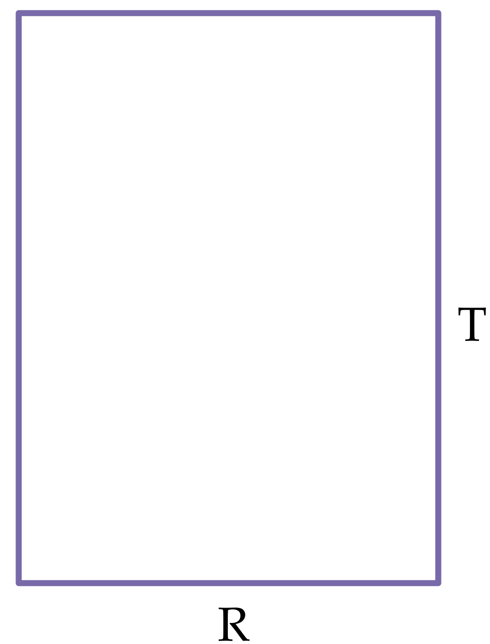
- Gedanken-experiment
  - Energy for the system that heavy quark and anti-quark are put with a separation R ?
  - Amplitude

$$\left\langle Q\bar{Q} \right| e^{-HT} \left| Q\bar{Q} \right\rangle = \frac{1}{Z}\int [dA_\mu] e^{-S+ig\oint_C dx_\mu A_\mu}$$

  - Potential

$$V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\left\langle \frac{1}{3}\mathrm{Tr}\,\mathrm{P}e^{ig\oint_C dx_\mu A_\mu} \right\rangle$$

    - P stands for the "path ordering"



Heavy quark potential

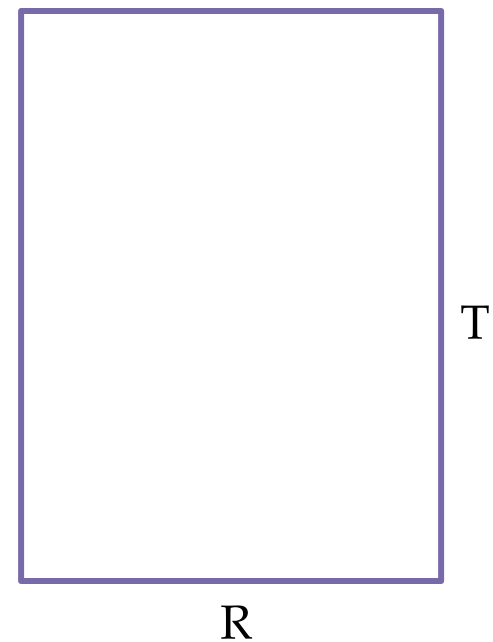
- In the lattice theory,
  - Given by a product of gauge links.

$$Z(C) = \left\langle \prod_C U \right\rangle$$

$$= \frac{\int [dU_\mu] \left(\prod_C U \right) e^{-S}}{\int [dU_\mu]\, e^{-S}}$$

    = "Wilson loop"

  - Integral over SU(N) for each gauge links
    = Integral over "gauge configurations"



Strong coupling expansion

- An expansion around β=6/g² = 0
  - Boltzman factor

$$e^{-S} = \prod_P e^{-\beta \mathrm{Tr}[UUUU]} \approx \prod_P \left[1 - \beta \mathrm{Tr}[UUUU]\right]$$

= No weight in the β=0 limit, completely random.

  - Formulae

$$\int [dU] = 1, \quad \int [dU] f(U) = \int [dU] f(U_0 U),$$

$$\int [dU] U_{ij} = 0, \quad \int [dU] U_{ij} U^\dagger_{kl} = \frac{1}{N} \delta_{il} \delta_{jk}$$

    - vanishes when only one *U* appears; non-zero when a pair of *U* and *U*⁺ appears.

Wilson loop

- Pull down P's from the action so that U and U^{+} makes a pair.

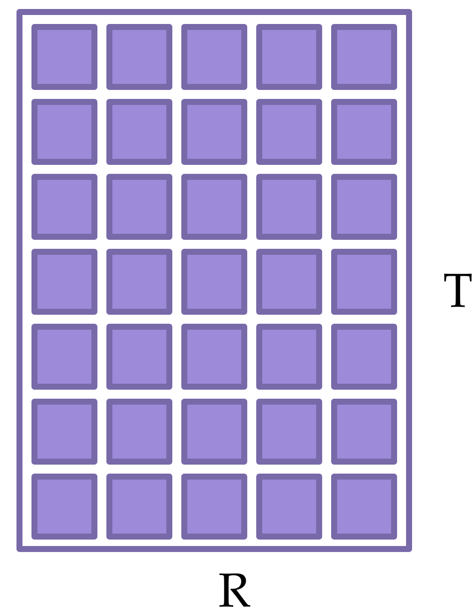
$$\left\langle \prod_C U \right\rangle \approx \left(\frac{1}{g^2 N} \right)^{RT}$$

  - Area law of the Wilson loop

- Potential

$$V(R) = \sigma R, \quad \sigma \sim \ln(g^2 N)$$

  - proportional to the distance = confinement
  - consequence of the random gauge configurations.

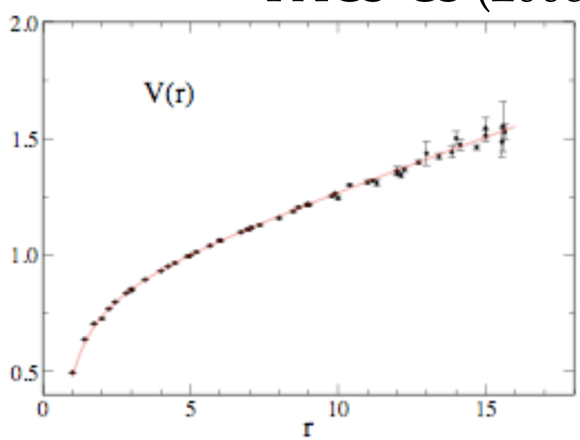


= Understanding of confinement

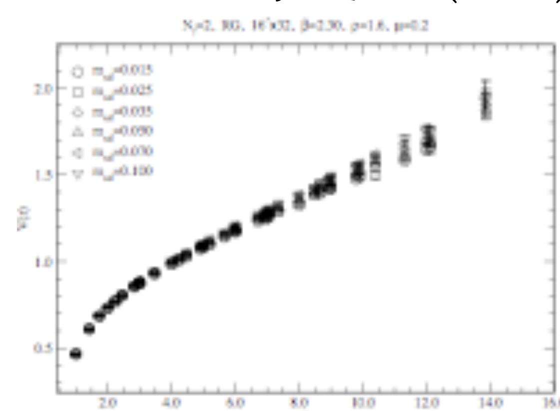
What is g^2?

- This is not the end of the story of confinement.
  - The coupling constant g^2 is the bare value. It goes to zero in the continuum limit (see below). Strong coupling expansion is not applicable.
  - Numerical study of the Wilson loop at weak couplings.
    - Linear-rising potential is certainly obtained.

PACS-CS (2008)



JLQCD (2006)



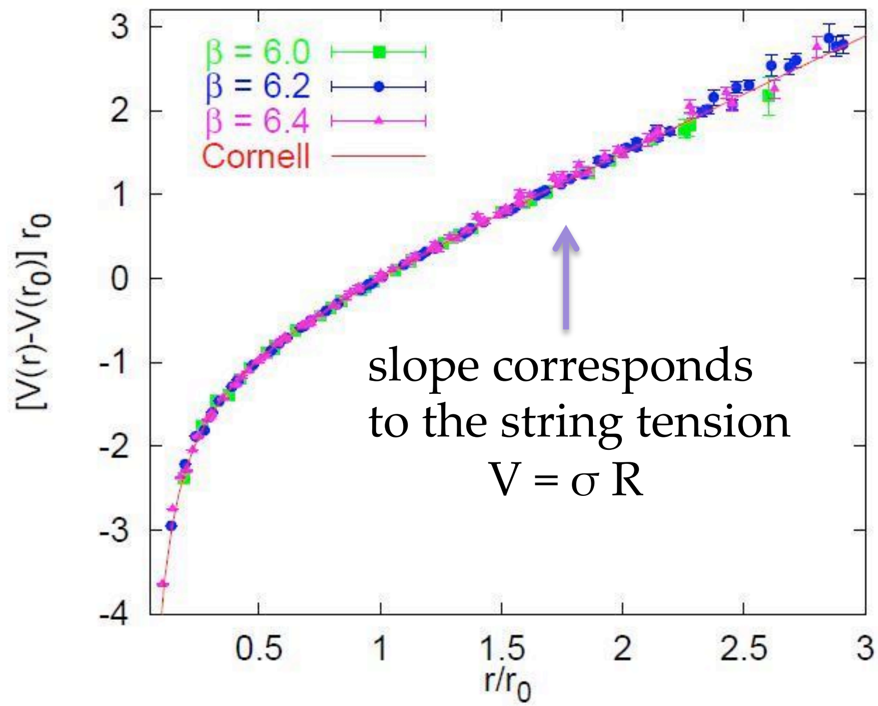
What is g^2?

- Pick a value of $\beta=6/g^2$, then …?

  = again, the question of renormalization group

$$g^2(a)$$

- Determined with some input, such as the string tension σ ~ (440 MeV)^2.
  - could be any other (dimensionful) quantities



What is g²?

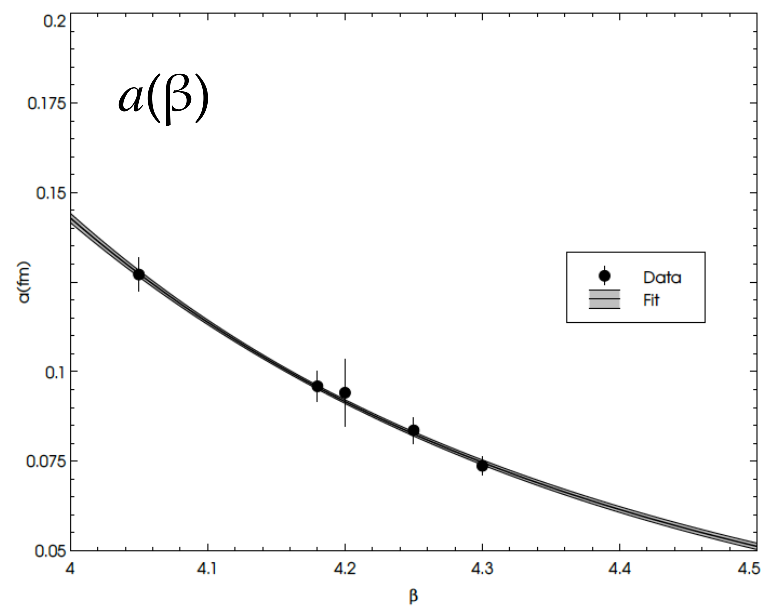
1. Pick a value of β=6/g²
2. Input a physical quantity

$$g^2(a)$$

3. Should depend on a according to RG

$$\beta(\alpha_s) \equiv -\mu \frac{d\alpha_s}{d\mu} = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \cdots$$

4. Take the continuum limit



$$a = c_0 f(g^2)(1 + c_2 \hat{a}(g)^2), \quad \hat{a}(g)^2 \equiv \frac{f(g^2)}{f(g^2=1)},$$

$$f(g^2) \equiv (b_0 g^2)^{-b_1/2b_0^2} \exp\left(-\frac{1}{2b_0 g^2}\right), \quad b_0 = \frac{11}{(4\pi)^2}, \quad b_1 = \frac{102}{(4\pi)^4},$$

2. Lattice gauge theory

2.2 Fermions

Doubling, chiral symmetry

Naïve discretization

- Continuum fermion action

$$S = \int d^4x \left[\bar{\psi}(x)\gamma_\mu \partial_\mu \psi(x) + m\bar{\psi}(x)\psi(x) \right]$$

  - Replace the derivative by a discrete difference

$$S^{\text{naive}} = a^4 \sum_{x,\mu} \bar{\psi}(x)\gamma_\mu \Delta_\mu \psi(x) + a^4 \sum_x m\bar{\psi}(x)\psi(x),$$

$$\Delta_\mu \psi(x) = \frac{1}{2a}\left(\psi(x+\hat{\mu}) - \psi(x-\hat{\mu})\right)$$

  - Easy to make it gauge invariant

$$\Delta_\mu \psi(x) = \frac{1}{2a}\left(U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\right)$$

Propagator

- Free field propagator

$$S(k) = \frac{1}{\frac{i}{a}\sum_\mu \gamma_\mu \sin(ak_\mu) + m} = \frac{-\frac{i}{a}\sum_\mu \gamma_\mu \sin(ak_\mu) + m}{\frac{1}{a^2}\sum_\mu \sin^2(ak_\mu) + m^2}; \quad k_\mu \in \left[-\frac{\pi}{a}, +\frac{\pi}{a}\right]$$

  - Physical mode is at k~(0,0,0,0), but other modes at k~(π/a,0,0,0), (0, π/a,0,0), (π/a, π/a,0,0) all contribute to the propagation. There are 2^d = 16 modes.
  - Each pole corresponds to a continuum fermion propagator

$$S(k) \to \frac{m - i\gamma_\mu^{(A)} p_\mu}{m^2 + p^2}, \quad k = \frac{\pi^{(A)}}{a} + p, \quad \gamma_\mu^{(A)} = \gamma_\mu \underbrace{\cos \pi_\mu^{(A)}}_{\pm 1}$$

    - π_A stands for each pole:

$$\pi^{(0)} = (0,0,0,0), \pi^{(1)} = (\pi,0,0,0), \ldots$$

$$\pi^{(12)} = (\pi,\pi,0,0), \ldots, \pi^{(123)} = (\pi,\pi,\pi,0), \ldots, \pi^{(1234)} = (\pi,\pi,\pi,\pi)$$

Doubler

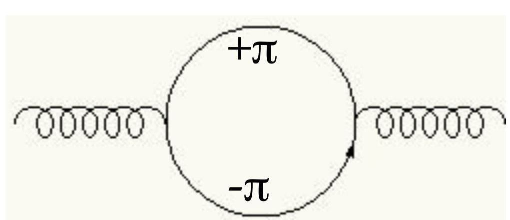
- All are equivalent
  - Unitary transformation (redefine the fermion field)

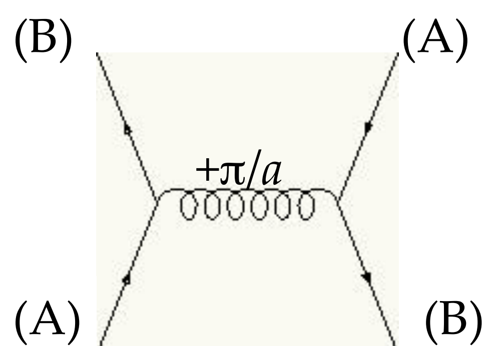
$$\{S^{(A)}\} = \left\{1, S_\rho, S_\rho S_\sigma, S_\rho S_\sigma S_\tau, S_1 S_2 S_3 S_4\right\}, \; S_\rho = i\gamma_\rho\gamma_5$$

  gives

$$S_\rho^\dagger \gamma_\mu S_\rho = \begin{cases} -\gamma_\mu & (\mu = \rho) \\ +\gamma_\rho & (\mu \neq \rho) \end{cases}$$

- Naïve lattice fermion leads to 16 equivalent continuum fermions.
  - Can we simply ignore? No!





What we are going to observe

- Naïve fermion has doublers. What to do?
  - Remove doublers, while breaking chiral symmetry = Wilson fermion
  - Live with doublers = staggered fermion
  - Remove doublers, while having a modified chiral symmetry = Ginsparg-Wilson fermions

  - Situation is summarized by the Nielsen-Ninomiya theorem.
  - Cannot win the both (no doubler and chiral symmetry) to have the correct axial anomaly.

Wilson fermion

- Add a mass term of O(1/a) to the doublers

$$m\sum_x \bar\psi(x)\psi(x) \to m\sum_x \bar\psi(x)\psi(x) + \frac{ar}{2}\sum_{x,\mu}\partial_\mu\bar\psi(x)\partial_\mu\psi(x)$$

$$= m\sum_x \bar\psi(x)\psi(x) + \frac{ar}{2}\sum_{x,\mu}\frac{1}{a^2}\left(\bar\psi(x+\hat\mu)-\bar\psi(x)\right)\left(\psi(x+\hat\mu)-\psi(x)\right)$$

$$= \left(m+\frac{4r}{a}\right)\sum_x \bar\psi(x)\psi(x) - \frac{r}{2a}\sum_{x,\mu}\left(\bar\psi(x+\hat\mu)\psi(x)+\bar\psi(x)\psi(x+\hat\mu)\right)$$

  - "mass" term $m+\frac{r}{a}\sum_\mu\left(1-\cos k_\mu a\right)$
  - doubler masses $m^{(A)} = m + 2n_A\frac{r}{a}$
    - n_A is the number of "π"
    - decouple in the continuum limit.

Wilson fermion

- The entire action

$$S = -\sum_{x,\mu}\left[\bar{\psi}(x)\frac{r-\gamma_\mu}{2}\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})\frac{r+\gamma_\mu}{2}\psi(x)\right] + M\sum_x \bar{\psi}(x)\psi(x)$$

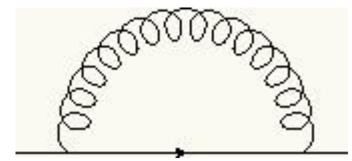
  - $M = ma+4\mathrm{r}$
- Then re-normalize

$$S = \sum_x \bar{\psi}(x)\psi(x) - \kappa\sum_{x,\mu}\left[\bar{\psi}(x)\left(r-\gamma_\mu\right)\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})\left(r+\gamma_\mu\right)\psi(x)\right]$$

  - $\kappa = 1/2M$
  - massless limit: $\kappa \to \kappa_c = 1/8\mathrm{r}$
- Chiral symmetry is lost. $\psi \to \exp(i\alpha\gamma_5)\psi, \bar{\psi} \to \bar{\psi}\exp(i\alpha\gamma_5)$
  - Wilson term remains even at m=0.

Problem of the Wilson fermion

- Chiral symmetry is recovered in the continuum limit. What is the problem, then?
  - Non-exact symmetry may be badly violated by quantum effect.
  - Ex.) Fermion self-energy
    - Continuum $$\int d^4k \frac{1}{k^2}\frac{\gamma_\alpha(-i\gamma_\mu k_\mu)\gamma_\alpha}{k^2} = 0$$



    - Naïve $$\int d^4k \frac{1}{\hat{k}^2}\frac{\gamma_\alpha(-i\gamma_\mu \bar{k}_\mu)\gamma_\alpha}{\bar{k}^2} = 0, \quad \bar{k}_\mu = \frac{1}{a}\sin(ak_\mu)$$
    - Wilson $$\int d^4k \frac{1}{\hat{k}^2}\frac{\gamma_\alpha\left[m+\frac{r}{2}\hat{k}^2 - i\gamma_\mu\bar{k}_\mu\right]\gamma_\alpha}{\left[m+\frac{r}{2}\hat{k}^2\right]^2+\bar{k}^2} \xrightarrow[m\to 0]{} \int d^4k \frac{1}{\hat{k}^2}\frac{2r\hat{k}^2}{\bar{k}^2+\frac{r^2}{4}(\hat{k}^2)^2}$$

Problem of the Wilson fermion

- There is an additive mass renormalization

  $$\delta m \approx \alpha_s \frac{1}{a}$$

  - Divergent in the continuum limit. Also in the higher order terms.
- How to define the quark mass?
  - One should decide some definition and measure it.
    - GMOR relation
    - Ward-Takahashi identity
  - Not unique.

Problem of the Wilson fermion

- Ward-Takahashi identity

$$\partial_\mu A_\mu^a(x) = 2m_q P^a(x)$$

  - Not satisfied for the Wilson fermion.
  - Require it to be satisfied and determine the quark mass.

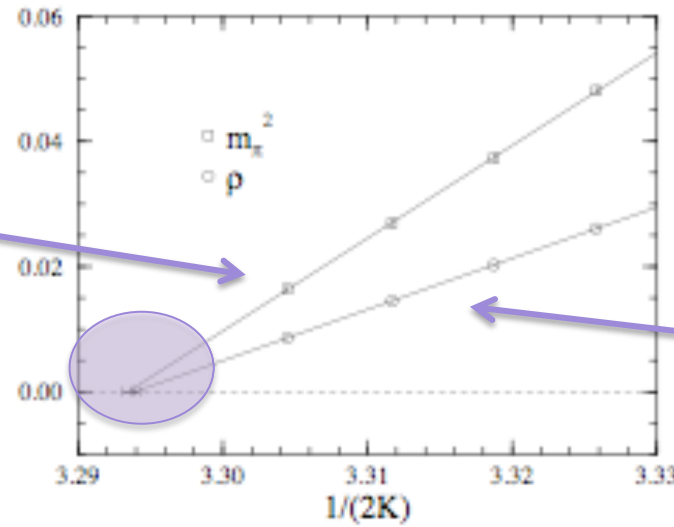
$$m_q \equiv \frac{\langle 0 | \partial_\mu A_\mu^a(x) | P^a \rangle}{\langle 0 | 2P^a(x) | P^a \rangle}$$

    - May and does depend on $|P\rangle$ and x, but decide to use one. Should converge in the continuum limit.

Chiral limit

- Achieved by a parameter tuning

GMOR
$m_\pi^2 \sim m_q$



4+ma

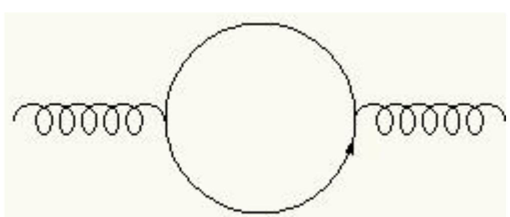
$$m_q \equiv \frac{\langle 0 | \partial_\mu A_\mu^a(x) | P^a \rangle}{\langle 0 | 2P^a(x) | P^a \rangle}$$

Looks like vanishing at the same point. This corresponds to the "chiral" limit. Could become a problem for precise calculations

Staggered fermion

= Essentially the same as the naïve fermion.

- There are doublers.
  - The number is reduced to 4 by eliminating redundant copies. The remaining 4 are intertwined.
  - Let's interpret them as up, down, strange and charm… Possible but too complicated. (Need non-degenerate masses. Intrinsic flavor-changing currents cause a lot of troubles.
  - Reduce by "hand"… = rooting.

 × 1/4

Or, introduce a fourth-root of the fermion determinant. Uncontrolled error may be induced.

(Rooted) staggered okay?

- Staggered fermion: 4 tastes per flavor, take a 4th-root.
  - Tastes are mixed at finite *a*. Rooting is non-trivial.
  - Triggered (painful) debates for years… not completely settled.

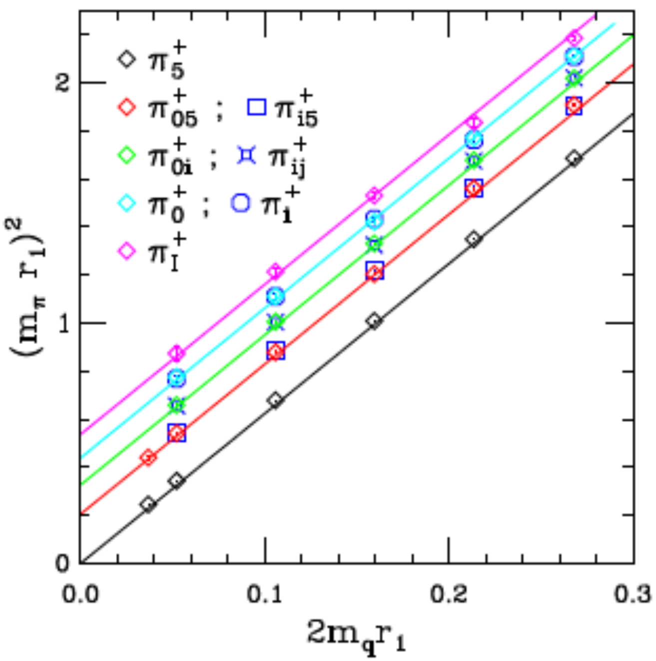
- My position:
  - No proof available, but probably okay in the continuum limit.
  - Practical issue = Are we close enough to continuum?
  - May depend on the quantity of interest (again). (Worst) Ex:
    - Typical size of violation: $a^2\Lambda^3$ ~ 10 MeV (when $1/a$=2 GeV)
    - Lowest-lying Dirac eigenvalue: $3/\Sigma V$ ~ 3 MeV (for $V=(3\text{ fm})^4$).

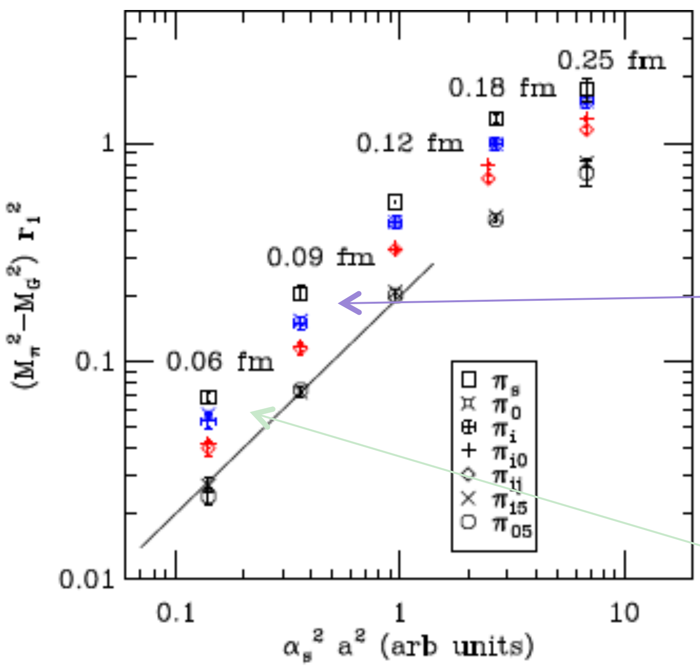
Low-modes are largely distorted. Effects on (many) physical quantities are non-trivial. Probably largest for pions.

Taste violation

- Seen in the data

From MILC (2007)





a ~ 0.09 fm:

When m_π=135 MeV, heaviest "pion" is 320 MeV.

a ~ 0.06 fm:

Reduced down to 210 MeV

For HISQ, ~ 1/3 of the above.

Modified chiral symmetry

- Nielsen-Ninomiya no-go theorem
  - No way to realize lattice chiral symmetry and non-doubling.
  - Indeed, this is necessary to realize the axial anomaly…, a deeper theoretical question.

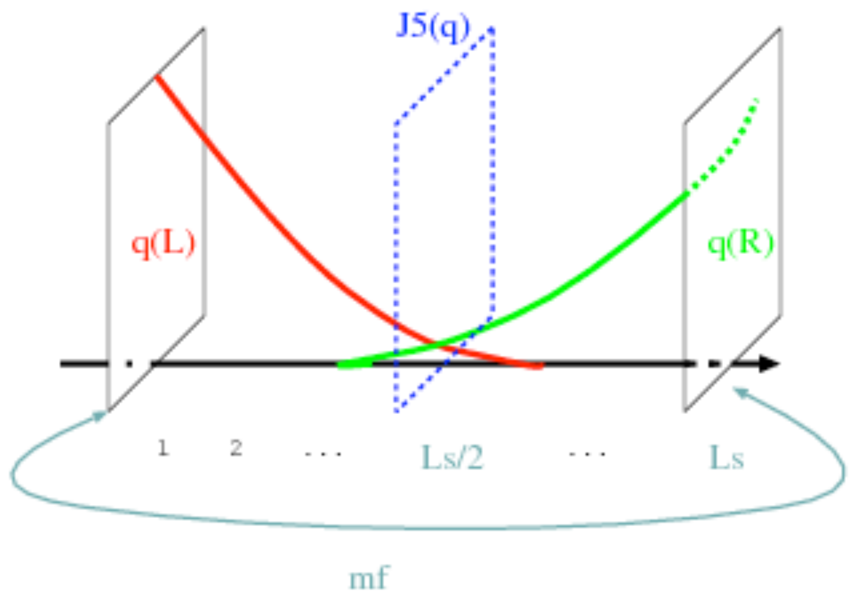
- Modify the "chiral symmetry" on the lattice

$$\delta\bar{\psi} = i\alpha\bar{\psi}\left(1-\frac{a}{2\rho}D\right)\gamma_5,\ \delta\psi = i\alpha\gamma_5\left(1-\frac{a}{2\rho}D\right)\psi$$

  - Go back to the ordinary definition at $a = 0$.
  - Related to the domain-wall and overlap fermions.

Domain-wall fermion

- Defined on 5D space
  - gauge field in the 5th direction is trivial.
  - design a mass term such that



DW/OV fermions

- Exact (but modified) chiral symmetry at finite *a* (in the limit of $L_s = \infty$)
  - Property of the Ginsparg-Wilson fermions that satisfy

$$D\gamma_5 + \gamma_5 D = \frac{a}{\rho} D\gamma_5 D$$

  - Ward-Takahashi identities are the same as in the continuum.
  - Axial-anomaly is reproduced.

- Drawback = numerical cost
  - 5D implementation.
  - Or, numerical approximation of sign function.

2. Lattice gauge theory

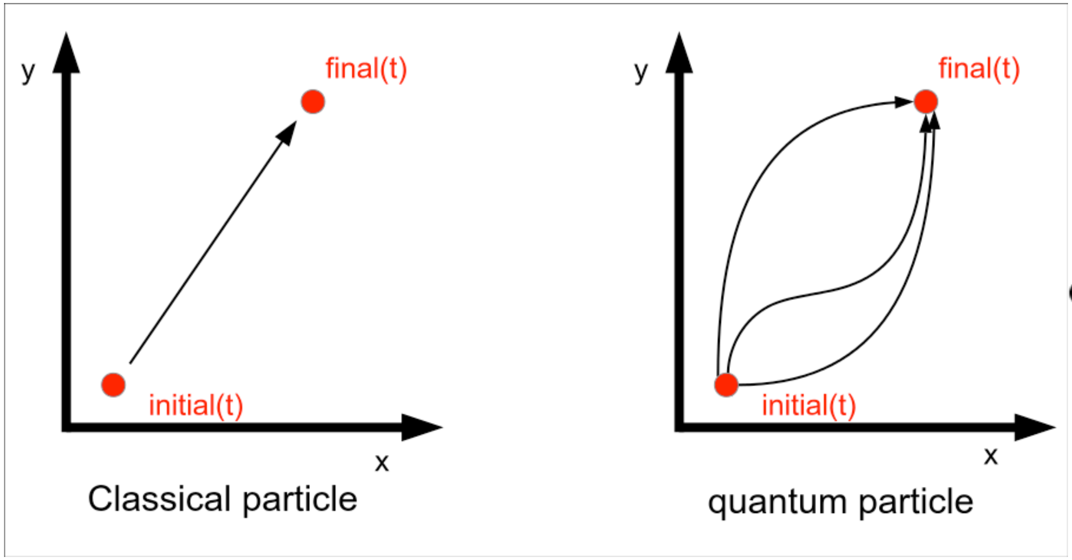
2.3 Computations

Path integral, Observables

Path integral formulation

- Actual calculation needs the path integral quantization

Evolution of a state:

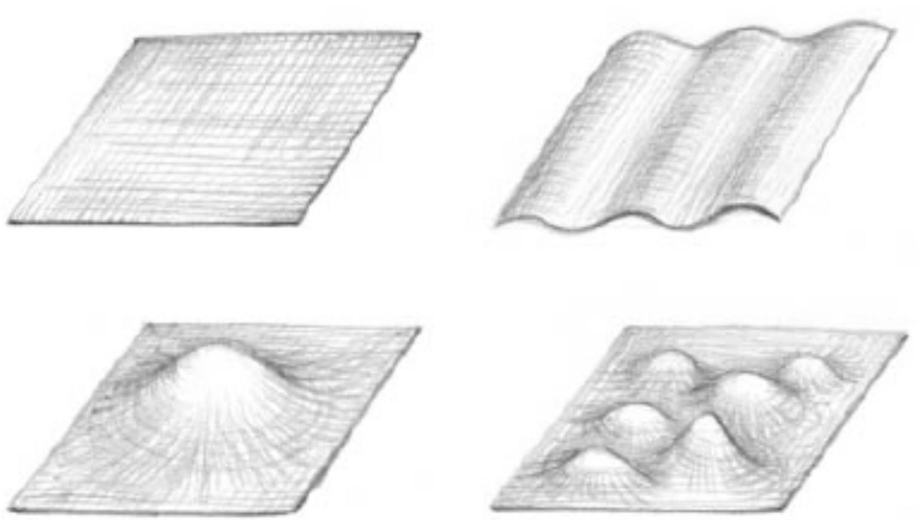


$$\exp\left(\frac{i}{\hbar}S(x_i, x_f)\right)$$

Sum the amplitudes corresponding to all possible paths.

Path integral formulation

- In quantum "field" theory, it is a sum over all possible fields:



$$Z = \int [d\phi] e^{iS}; \; S = \int d^4 x \mathcal{L}$$

- There is an "amplitude" e^{iS} for each field "configuration"
- Sum the amplitudes over all possible configurations.

Okay, let's carry out!

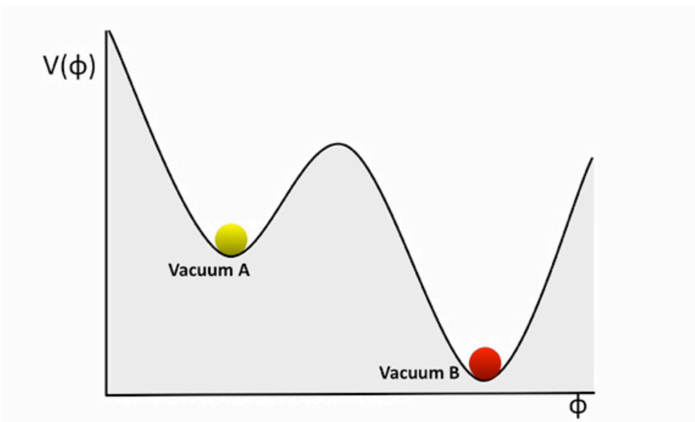
- Sounds easy?
  - Super-multiple integral…, actually infinitely many!

$$Z = \int [d\phi] e^{iS}; \;\; S = \int d^4 x \mathcal{L}$$

- Possible when the integral is known = Gaussian
  - Free field theory: $S \sim \phi^2$
  - Expansion around this simplest case = perturbation theory
  - Good approximation if the reality is sufficiently "free".

What is perturbation theory?

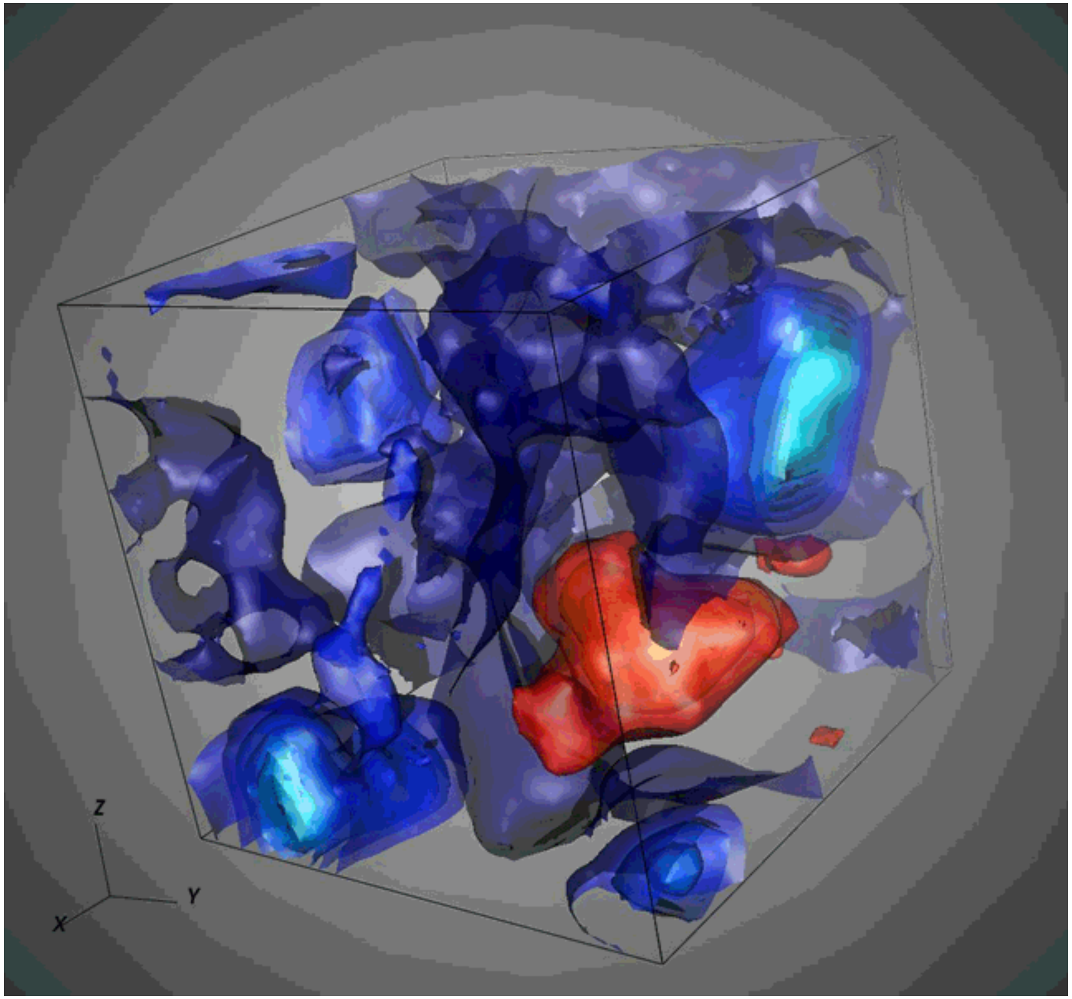
- Reduces to harmonic oscillator:
  - When the potential is complicated, try to expand around its bottom.



  - Good approximation if the field actually fluctuates around there.
  - If the fluctuation is bigger…, no way.

What is the *vacuum*?

- In QED,
  - $F_{\mu\nu}=0$ is the vacuum.
  - Photon is an excitation from there.
- In QCD,
  - More fluctuations. The vacuum is determined as the minimum of the "effective action", which is the free energy in the language of statistical mechanics.
  - But, not completely random either.
  - Particles represent the excitations on this "vacuum".


Z
Y
X

Correspondence

Statistical mechanics

- partition function; Hamiltonian

$$Z=\int[d\phi]e^{-H/T};\ H\sim\int d^3x\mathcal{H}$$

Quantum field theory

- partition function; action

$$Z=\int[d\phi]e^{iS};\ S=\int d^4x\mathcal{L}$$

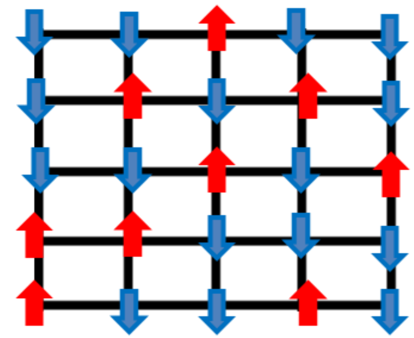
- After the Wick rotation, it is made Euclidean

$$Z=\int[d\phi]e^{-S_E};\ S_E=\int d^4x\mathcal{L}$$

Monte Carlo: a simple example

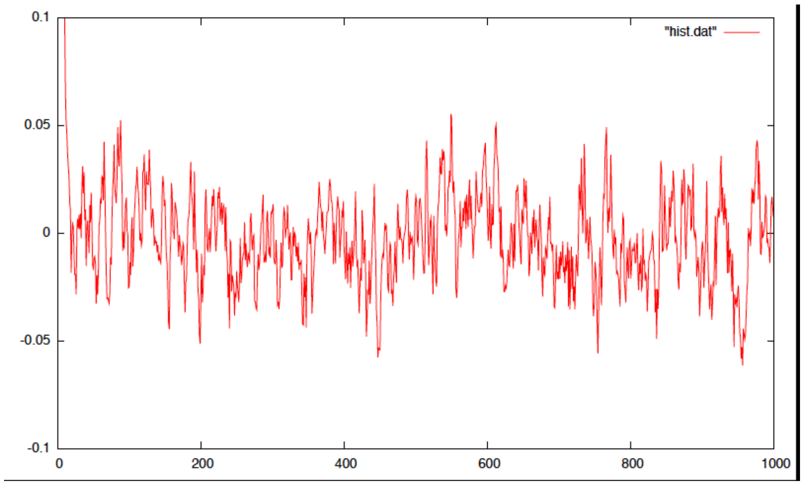
Ising model

$$Z = \sum_{\{s_i\}} \exp\left[-H\{s_i\} / T \right], \quad H\{s_i\} = -J \sum_{\{i,j\} \in n.n.} s_i s_j$$



How does the spontaneous magnetization emerge?

$$M = \frac{1}{L^2} \sum_i s_i$$



Monte Carlo method

Basic idea:

$$Z = \sum_{\{s_i\}} \exp\left[-H\{s_i\}/T\right], \quad H\{s_i\} = -J \sum_{\{i,j\}\in n.n.} s_i s_j$$

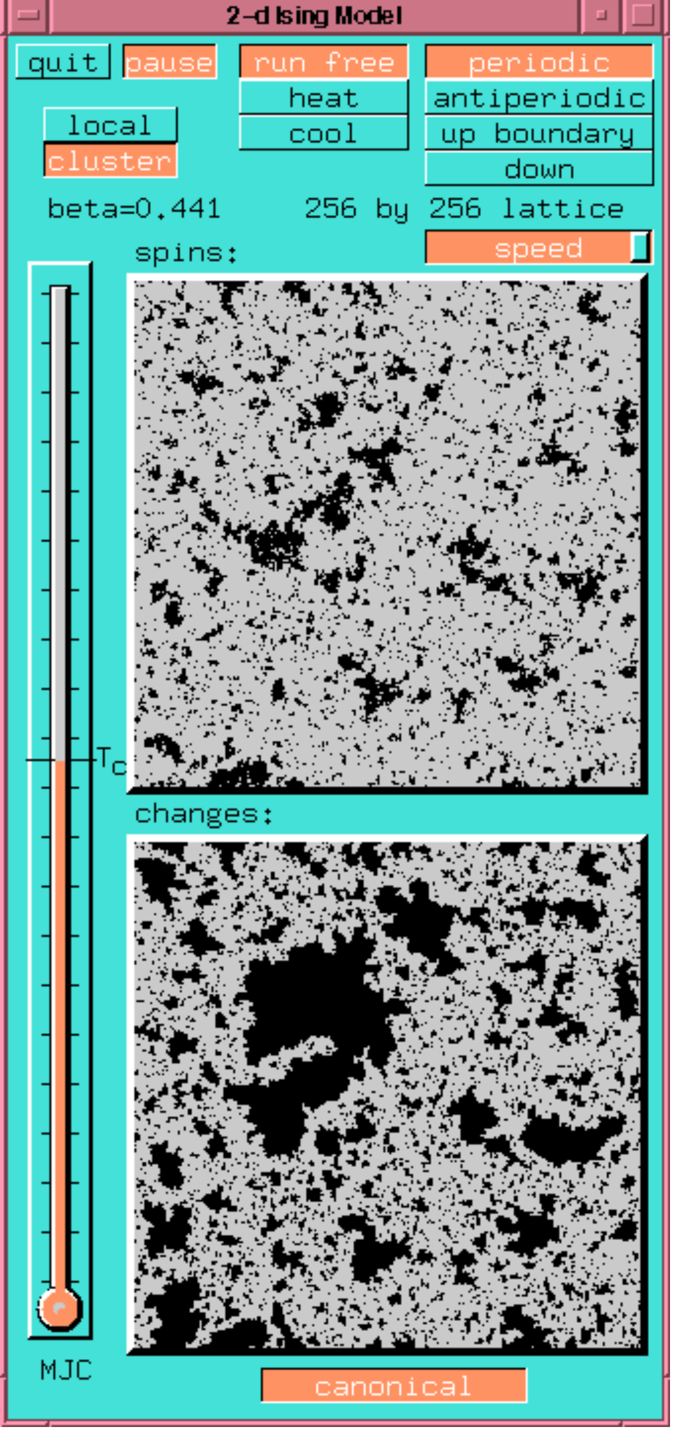
- The number of terms = 2^(2L^2). For L=100, it is 2^{20000}~10^{2000}. Impossible.
- Only some limited terms contribute to the sum:
  - T = 0: only those giving the minimum H{s_i}.
  - T = ∞: completely random.
- Pickup the relevant configurations only = MC

Procedure

Without proof…

1. Starting from some initial config $\{s_i\}$, generate the next config $\{s_i'\}$ with rand.
2. Calculate the initial and final Hamiltonians H, H'
3. Metropolis accept/reject
   1. If H'<H, accept the new config $\{s_i'\}$
   2. If H'>H, accept with a probability exp(-(H'-H)/T)
4. Goto 1 and Repeat until stabilized.

- Expectation value <M> is obtained as an average over the configs thus generated.



Demo

xtoys: written by Mike Creutz

Can you tell
- Magnetization?
- Correlation length?
- Their temperature dependence?

Let's go back to QCD

- Too hard to evaluate
  - Determinant of a large matrix. Needs to obtain all the eigenvalues ~ N[3]

$$\det(D[U]+m)=\prod_k (m+i\lambda_k[U])$$

- Rewrite in favor of bosons

non-local action

$$Z=\int[dU]\det(D[U]+m)^2 e^{-S_g}$$
$$=\int[dU][d\phi]e^{-S_g-\phi^\dagger(D[U]+m)^{-2}\phi}=\int[dU][d\phi]e^{-S_g-|(D[U]+m)^{-1}\phi|^2}$$

  - Reduces to the problem of matrix inversion. Hard, but more tractable.

Matrix inversion

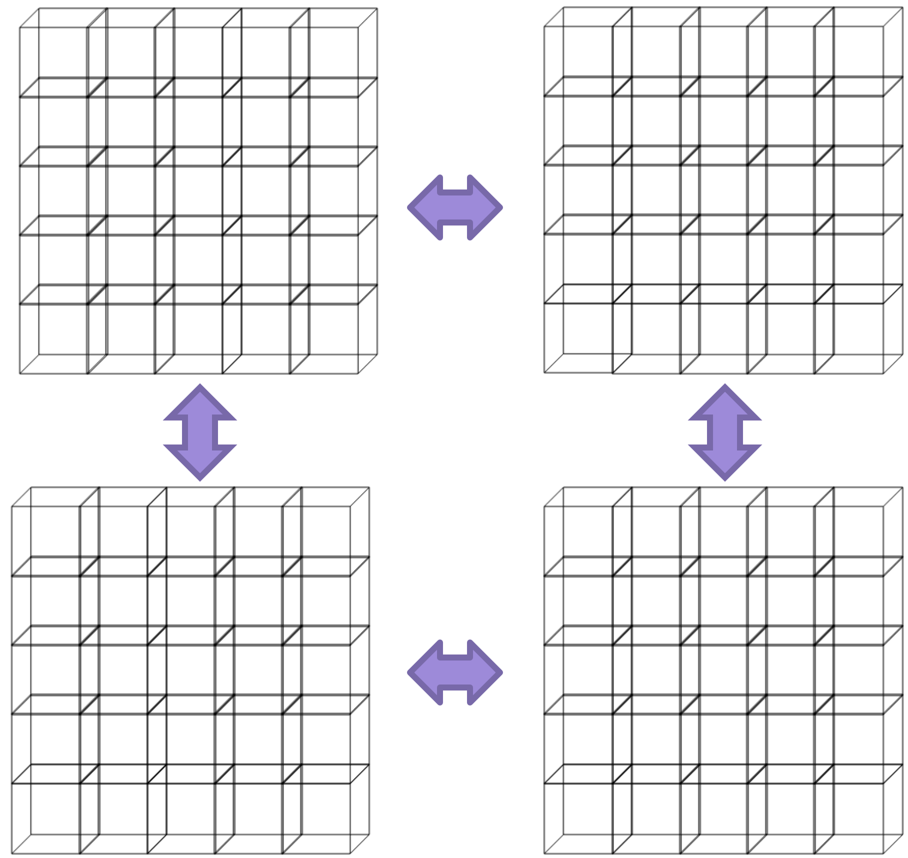
- Most time-consuming part in the LQCD calculations

$$\left(D[U]+m\right)x=b$$

  - $D[U]$: a 4D diffusion-like operator (typically nearest-neighbor)
  - In some cases, use 5D implementation for theoretical virtue
- 4D lattices:
  - Typical size: 64^3x128 x3(color)x4(spinor) = 400 M
  - 1 vector = 7 GB
- Iterative solver:
  - Conjugate Gradient (CG): typically 1,000-10,000 iterations per solve

Big computing

- Parallel computing
  - Conceptually straightforward. Each node is responsible for a small sub-lattice.
  - Not "easy" in practice.
- Code development
  - CPS, Chroma, MILC, …
  - QMP, QDP, QUDA, …
  - Bagel, BFM
  - openQCD
  - Bridge++, Iroiro++



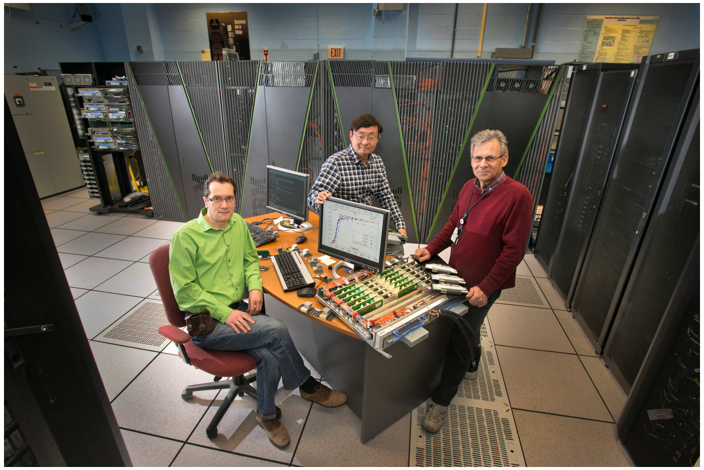
Supercomputer

- K computer (RIKEN Kobe)
  - Peak 11.3 Pflops (2011~)
  - Fujitsu SPARC64 VIIIfx

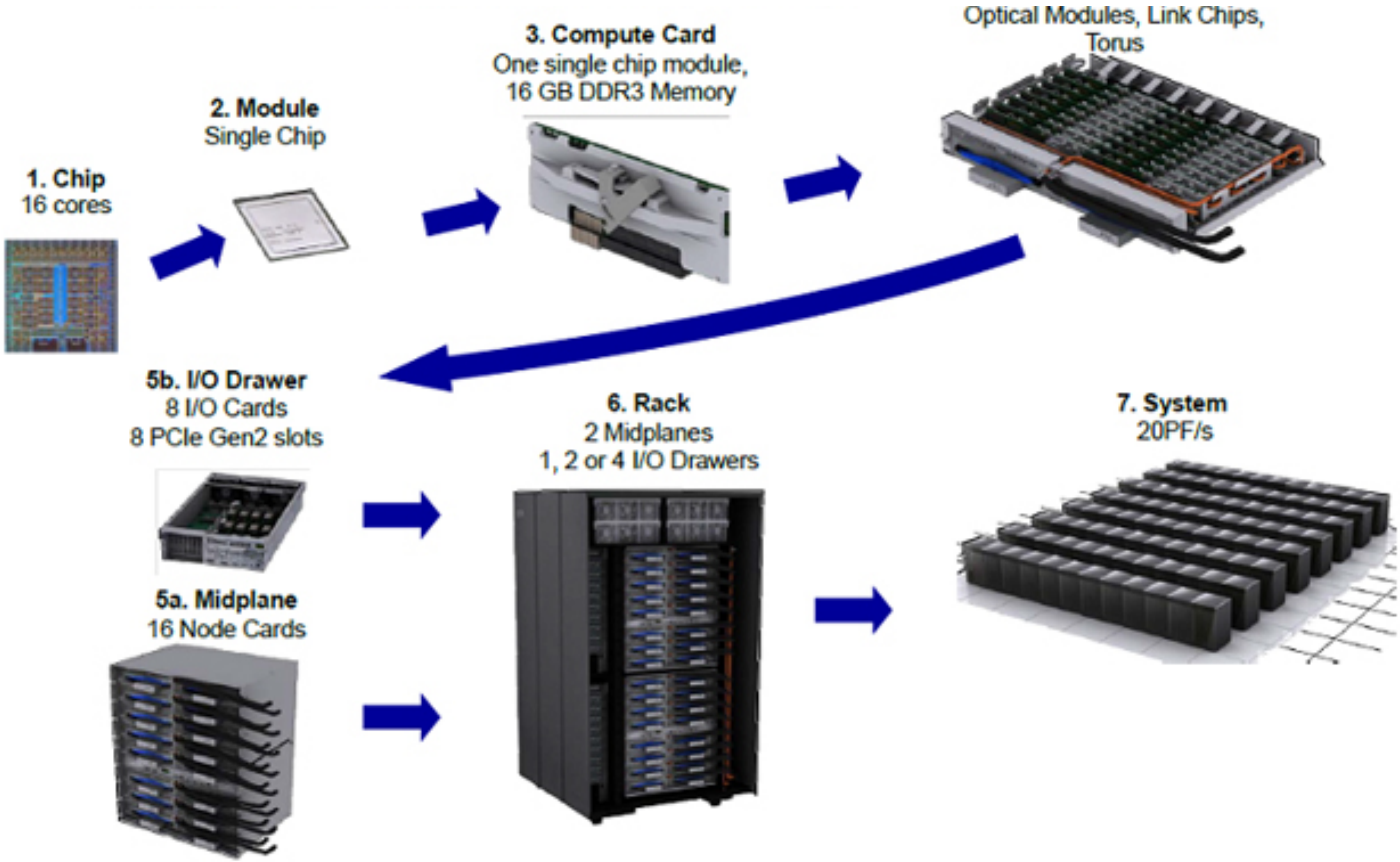
  - General purpose (life, environment, material, etc). Running QCD, too
  - Next generation project (Flagship 2020) has been launched. Aims at building a general purpose exascale machine by 2020.

Supercomputer

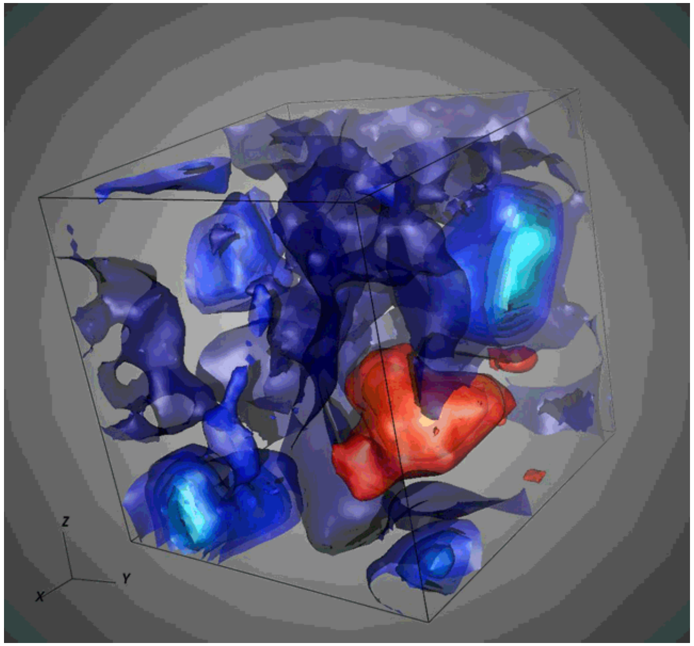
- IBM Blue Gene /Q
  - at LLNL, ANL, RIKEN/BNL, Julich, CINECA, Edinburgh, KEK, … are intensively used by LQCD.



Blue Gene /Q


1. Chip
16 cores
2. Module
Single Chip
3. Compute Card
One single chip module,
16 GB DDR3 Memory
Optical Modules, Link Chips,
Torus
5b. I/O Drawer
8 I/O Cards
8 PCIe Gen2 slots
5a. Midplane
16 Node Cards
6. Rack
2 Midplanes
1, 2 or 4 I/O Drawers
7. System
20PF/s

QCD vacuum?



Accumulation of near-zero eigenmodes of quarks leads to

- Chiral condensate

$$\langle \bar{q}q \rangle \neq 0$$

- Order parameter of the spontaneous chiral symmetry breaking.

Dirac eigenmodes

- Eigen equation $Du_\lambda = \lambda u_\lambda$
  - Fermion propagator $S(x,y) = -\sum_\lambda \frac{u_\lambda(x) u_\lambda^\dagger(y)}{m+\lambda}$
  - Chiral condensate

$$-\langle \bar{q}q \rangle = \int d^4x \,\mathrm{Tr}\left[S(x,x)\right] = \sum_\lambda \frac{1}{\lambda + m} = \sum_{\mathrm{Im}\lambda > 0} \frac{2m}{|\lambda|^2 + m^2}$$

    - Vanishes if m→0 is taken first. To obtain correctly, the limits must be in the order of V→0 and m→0 (thermodynamical limit)

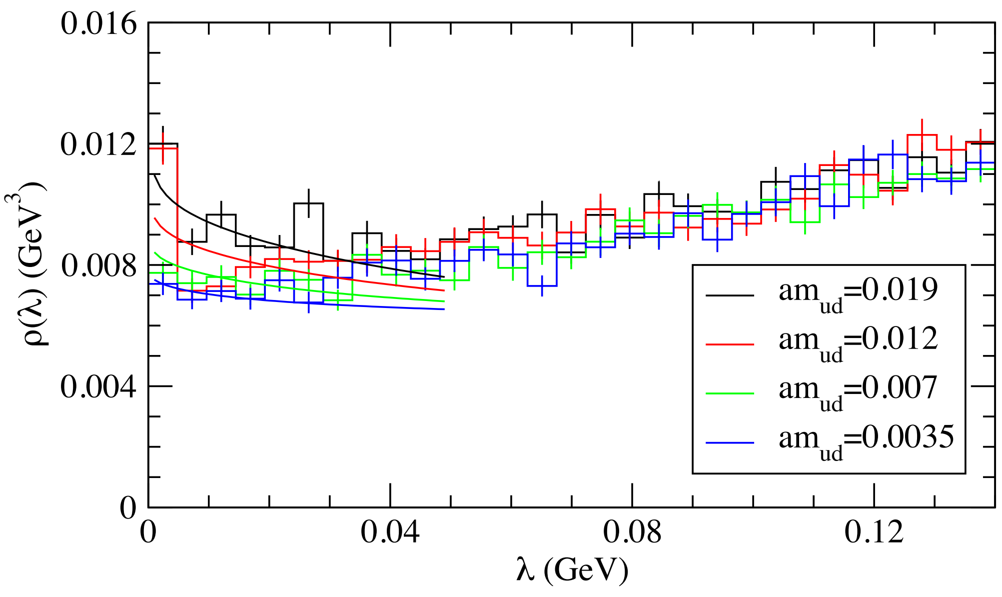
$$-\langle \bar{q}q \rangle = \int_0^\infty d\lambda\, \rho(\lambda) \frac{2m}{\lambda^2 + m^2} = \pi\rho(0)$$

ρ(λ): eigenvalue density

Banks-Casher relation： accumulation of low-lying modes

Dirac spectrum

Eigenvalue distribution of $\not{D}$



$$\Sigma = (270.0 \pm 4.9\ \text{MeV})^3$$

Physical quantities

- Two-point correlation function

$$\langle O_\Gamma(x) O_{\Gamma'}(y) \rangle = \frac{1}{Z} \int [dU] O_\Gamma(x) O_{\Gamma'}(y) e^{-S}$$

  - Ex. Fermion bilinear

$$P^a(x) = \bar{q}(x)\gamma_5 \frac{\tau^a}{2} q(x),\ A_\mu^a(x) = \bar{q}(x)\gamma_\mu\gamma_5 \frac{\tau^a}{2} q(x),$$

  - Two point function contains the info of all the intermediate states

$$\langle 0 | P^a(x) P^{a\dagger}(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \sum_n \left| \langle 0 | P^a(0) | P^{(n)}(p) \rangle \right|^2 \frac{e^{ip(x-y)}}{(m^{(n)})^2 + p^2}$$

  - No pole associated with the particle propagator exists on the Euclidean lattice, but obtain it assuming analyticity.

Ground state

- Rely on the analyticity
  - Look at the time correlation after specifying the spatial momentum.

$$C^{(2)}(t) \sim \int_{-\pi/a}^{+\pi/a} \frac{dp_0}{2\pi} \frac{e^{ip_0 t}}{m^2 + p_0^2 + \mathbf{p}^2} = \frac{1}{2E(\mathbf{p})} e^{-E(\mathbf{p})t}$$

  - The lowest energy states dominate at long separations.

$$\int d^3x \langle 0 | P^a(x) P^{a\dagger}(0) | 0 \rangle = \sum_n \frac{\left| \langle 0 | P^a(0) | P^{(n)}(p) \rangle \right|^2}{2E^{(n)}(\mathbf{0})} e^{-E^{(n)}(\mathbf{0})t}$$

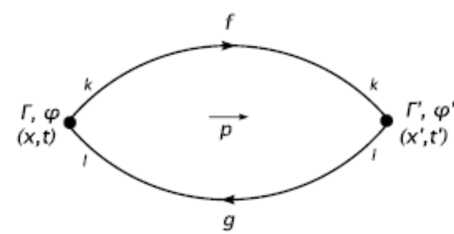
$$\xrightarrow[t\to\infty]{} \frac{\left| \langle 0 | P^a(0) | P^{(0)}(p) \rangle \right|^2}{2E^{(0)}(\mathbf{0})} e^{-E^{(0)}(\mathbf{0})t}$$

  - Ground state energy (mass) and matrix element is obtained.

Calculation of the correlator

- Can be rewritten using the quark propagators:

$$\left\langle O_\Gamma(x) O^\dagger_{\Gamma'}(y) \right\rangle = \left\langle \mathrm{Tr}\left[\Gamma S(x,y) \Gamma' S(y,x) \right] \right\rangle$$



  - Quark propagator is obtained by solving

    $$[D+m]S(x,y) = \delta_{x,y}$$

  - One may also use the relation $S(y,x) = \gamma_5 S^\dagger(x,y) \gamma_5$

- Connected two-point function (meson and baryon)
  - Fermion matrix inversion for each component (3x4=12)
  - Starts from a given point of space-time, and ends at any point.

Operators

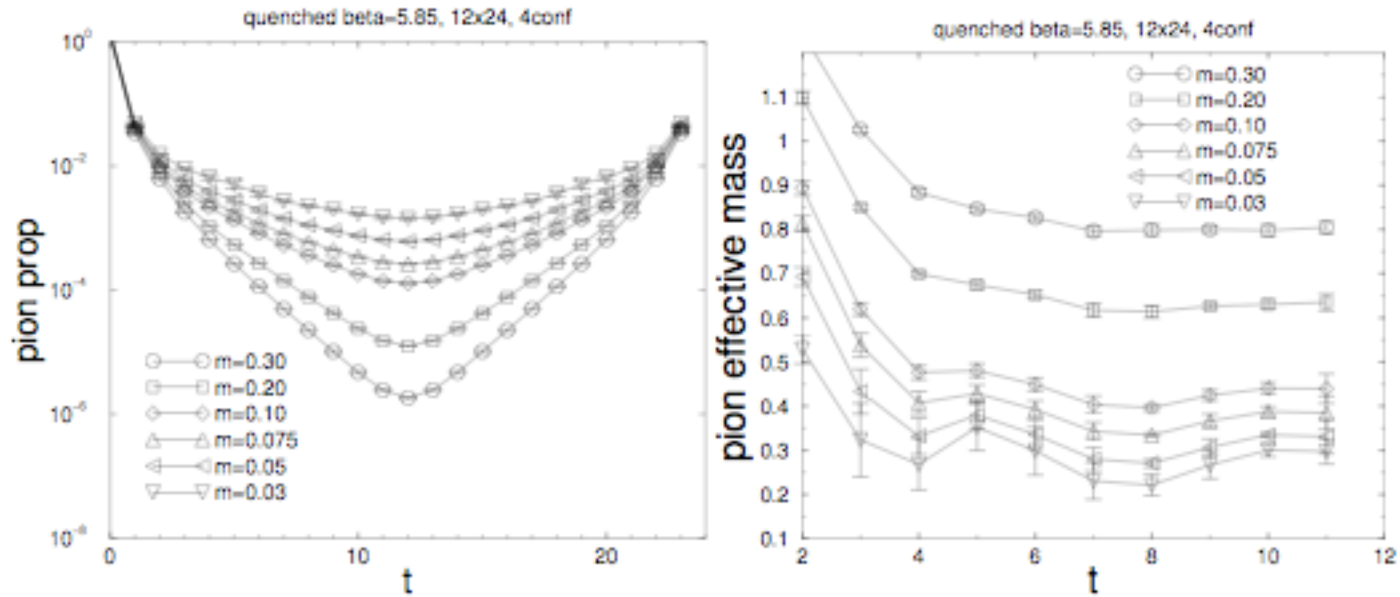
- Arbitrary as far as it has the same quantum number with that of the particle of interest.

$n\,^{2s+1}\ell_J$	J^{PC}	$I=1$ $u\bar{d},\ \bar{u}d,\ \frac{1}{\sqrt{2}}(d\bar{d}-u\bar{u})$	$I=\frac{1}{2}$ $u\bar{s},\ d\bar{s};\ \bar{d}s,\ -\bar{u}s$	$I=0$ f'	$I=0$ f	
$1\,^1S_0$	0^{-+}	π	K	η	$\eta'(958)$	γ_5
$1\,^3S_1$	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	γ_i
$1\,^1P_1$	1^{+-}	$b_1(1235)$	$K_{1B}{}^\dagger$	$h_1(1380)$	$h_1(1170)$	
$1\,^3P_0$	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$	I
$1\,^3P_1$	1^{++}	$a_1(1260)$	$K_{1A}{}^\dagger$	$f_1(1420)$	$f_1(1285)$	$\gamma_5\gamma_i$

  - In many cases, only the S wave states are considered. The P wave states are very noisy.
  - Spatially extended operators (smearing) is used to enhance the ground state signal.

Example

- Data look like this.



  - Effective mass E(t) defined as $E(t) = -\ln\dfrac{C(t+1)}{C(t)} \rightarrow E^{(0)}$

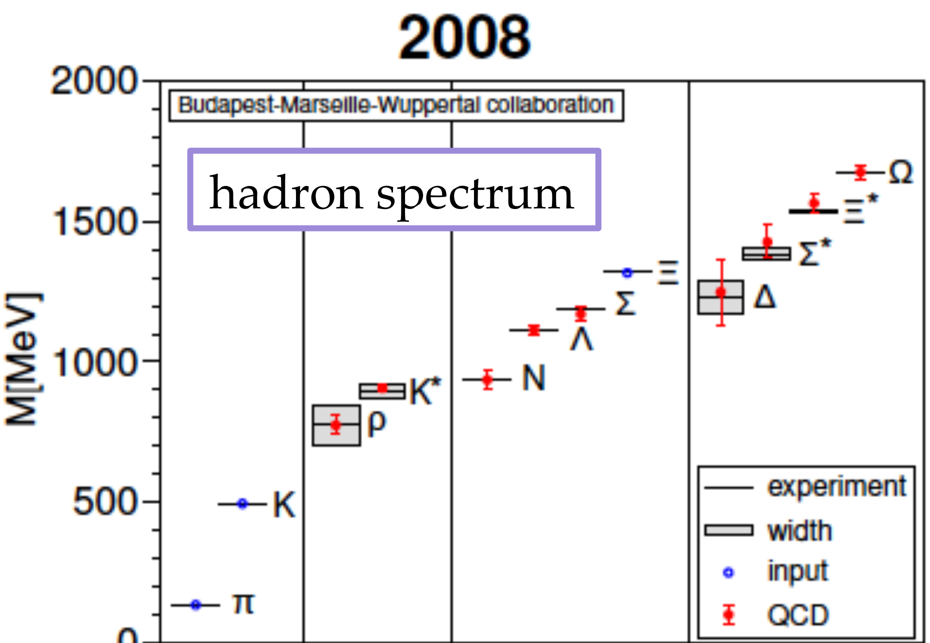
INPUT to LQCD

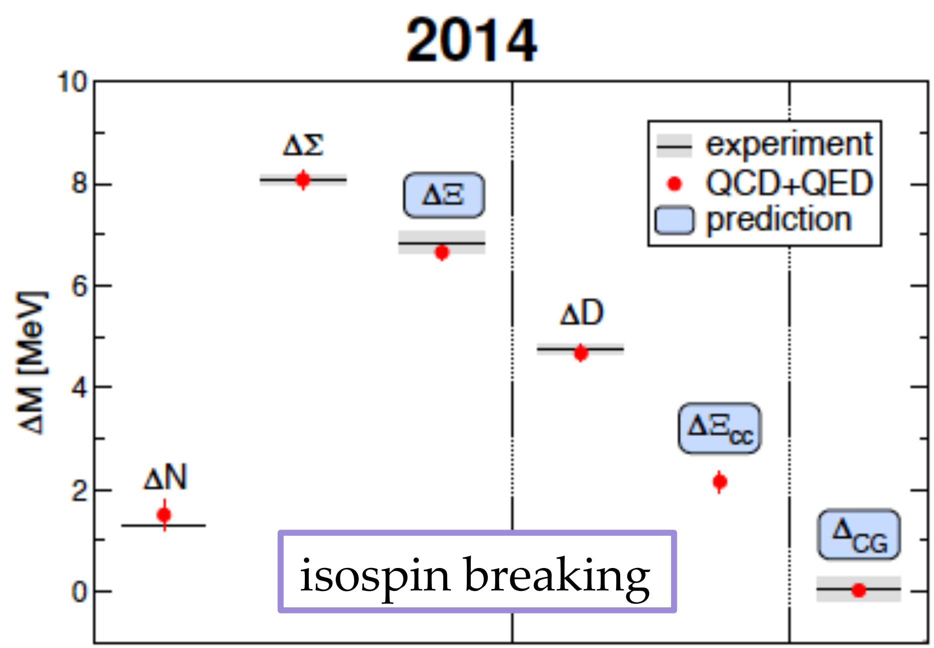
Parameters in QCD

- Strong coupling constant $\alpha_s(\mu)$
  - Fix the correspondence between the scale and coupling.
  - β is the relevant parameter to control the lattice spacing a.
- Light quark masses m_u, m_d, m_s
  - up and down are often assumed to be degenerate.
  - Tuned to reproduce π and K meson masses.
- Heavy quark masses m_c, m_b
  - Usually not in the sea, but changing.
  - Tuned to reproduce J/ψ and Υ masses.

All the other quantities are OUTPUT.

Hadron spectrum





Budapest-Marseille-Wuppertal collaboration, Science (2008, 2015)

X. Low-energy QCD

Chiral symmetry breaking

- In the QCD vacuum, chiral symmetry is broken.
  - Flavor $SU(3)_L xSU(3)_R \to SU(3)_V$
  - Non-zero chiral condensate $\langle \bar{q}q \rangle$
  - Nambu-Goldstone bosons (pion, kaon, η) nearly massless; in practice massive due to non-zero m_q.
    - Flavor-singlet axial U(1) is special, due to anomaly. η' is substantially heavier.
  - Other hadrons have a mass of $O(\Lambda_{QCD})$

  - Low energy effective theory for pions (and K, η) can be constructed = chiral perturbation theory (ChPT, χPT).

PCAC relation

- Partially Conserved Axial Current (PCAC)
  - From the QCD Lagrangian,

$$A_\mu = \bar{u}\gamma_\mu\gamma_5 d,$$

$$\partial_\mu A^\mu = (m_u + m_d)\bar{u}\gamma_5 d$$

  - The axial current may annihilate pion to the vacuum; Lorentz invariance restricts its form.

$$\langle 0|A_\mu(0)|\pi(p)\rangle = i f_\pi p_\mu,$$

$$\langle 0|\partial_\mu A^\mu(0)|\pi(p)\rangle = f_\pi m_\pi^2;$$

$$\partial_\mu A^\mu(x) = f_\pi m_\pi^2 \phi_\pi(x)$$

  - f_π is called the pion decay constant.
  - Can be measured from the leptonic decay $\pi\to\mu\nu$.

$$f_\pi = 131\ \text{MeV}$$

  - Its analog for kaon is f_K.

$$f_K = 160\ \text{MeV}$$

$\phi_\pi(x)$: operator to create a pion.

- Gell-Mann-Oakes-Renner (GMOR) relation (1968)

$$(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = -f_\pi^2 m_\pi^2 \left\{1 + O(m_\pi^2)\right\}$$

  - Chiral symmetry is broken = Non-zero chiral condensate $\langle \bar{q}q \rangle$
  - Pion mass squared is proportional to quark mass

$$m_\pi^2 = B_0(m_u + m_d) + O(m_q^2)$$
$$= \frac{-2\langle \bar{q}q \rangle}{f_\pi^2}(m_u + m_d) + O(m_q^2)$$

  - Also for kaons,

$$m_{K^+}^2 = B_0(m_u + m_s) + O(m_q^2), m_{K^0}^2 = B_0(m_d + m_s) + O(m_q^2),$$
$$m_\eta^2 = \frac{1}{3}B_0(m_u + m_d + 4m_s) + O(m_q^2),$$

  - Quark mass ratios can be predicted up to O(m_q^2).

Chiral Lagrangian

- Low energy effective lagrangian is developed assuming
  - Spontaneous breaking of chiral symmetry
  - Pion (and kaon, eta) to be the Nambu-Goldston boson
- In the low energy regime, pions are the only relevant dynamical degrees of freedom.

$$L_2 = \frac{f^2}{4}\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{\Sigma}{2}\mathrm{Tr}\left(mU^\dagger + Um^\dagger\right),$$

$$U = \exp\left(\frac{i\tau^a\pi^a}{f}\right)$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

  - Given by a non-linear sigma model.
  - Provides a systematic expansion in terms of m_π^2, p^2; the leading order is given above.

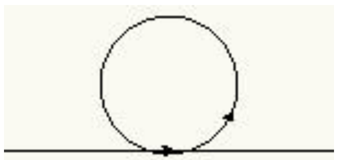
- Expansion in the pion field gives

$$L_2 = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a - \frac{m_\pi^2}{2}\pi^a\pi^a + \frac{m_\pi^2}{24 f^2}(\pi^a\pi^a)^2$$
$$+\frac{1}{6f^2}\left[(\pi^a\partial_\mu\pi^a)(\pi^b\partial^\mu\pi^b) - (\pi^a\pi^a)(\partial_\mu\pi^b\partial^\mu\pi^b)\right] + \ldots$$

  - Pion mass is obtained as $m_\pi^2 = 2B_0 m$
  - A chain of interaction terms: 4π, 6π, etc.
- Loop corrections are calculable.
  - Pick up a factor of $(m_\pi/4\pi f)^2$ or $(\boldsymbol{p}/4\pi f)^2$
  - Counter terms must also be added at order $(m_\pi/4\pi f)^2$ or $(\boldsymbol{p}/4\pi f)^2$
    - introduce the low energy constants (LECs): L_1~L_{10} at the one-loop level

One-loop example

- Pion self-energy



$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} = \frac{1}{(4\pi)^2}\left[\Lambda^2 + m^2 \ln \frac{m^2}{\Lambda^2 + m^2}\right]$$ Cutoff regularization

$$= \frac{m^2}{(4\pi)^2}\left(\frac{2}{\varepsilon} + \gamma - \ln 4\pi + \ln \frac{m^2}{\mu^2} - 1\right)$$ Dimensional reg

  - Log dependence $m^2\ln(m^2)$: called the chiral logarithm.
  - Comes from the infrared end of the integral = long distance effect of (nearly massless) pion loop.
  - Counter terms are necessary in order to renormalize the UV divergence.
  - After subtracting the UV divergences

$$m_\pi^2 = 2B_0 m_q \left[1 + \frac{1}{2}\frac{m_\pi^2}{(4\pi f)^2} \ln \frac{m_\pi^2}{\mu^2} + (\text{const}) \times \frac{m_\pi^2}{(4\pi f)^2} + O(m_\pi^4)\right]$$

Counter terms

- At the order $(m_\pi/4\pi f)^2$ or $(p/4\pi f)^2$, there are 10 possible counter terms
  - 10 new parameters, L_1~L_{10} = low energy constant at NLO
    c.f. 2 parameters at LO: Σ and f.
  - Depends on how one renormalizes the UV divergence, just as in the small coupling perturbation. L_1~L_{10} depends on the renormalization scale μ.
  - Once these parameters are determined (e.g. from pion scattering data), one can predict other quantities.

  - Lattice QCD may be used to *calculate* these parameters.

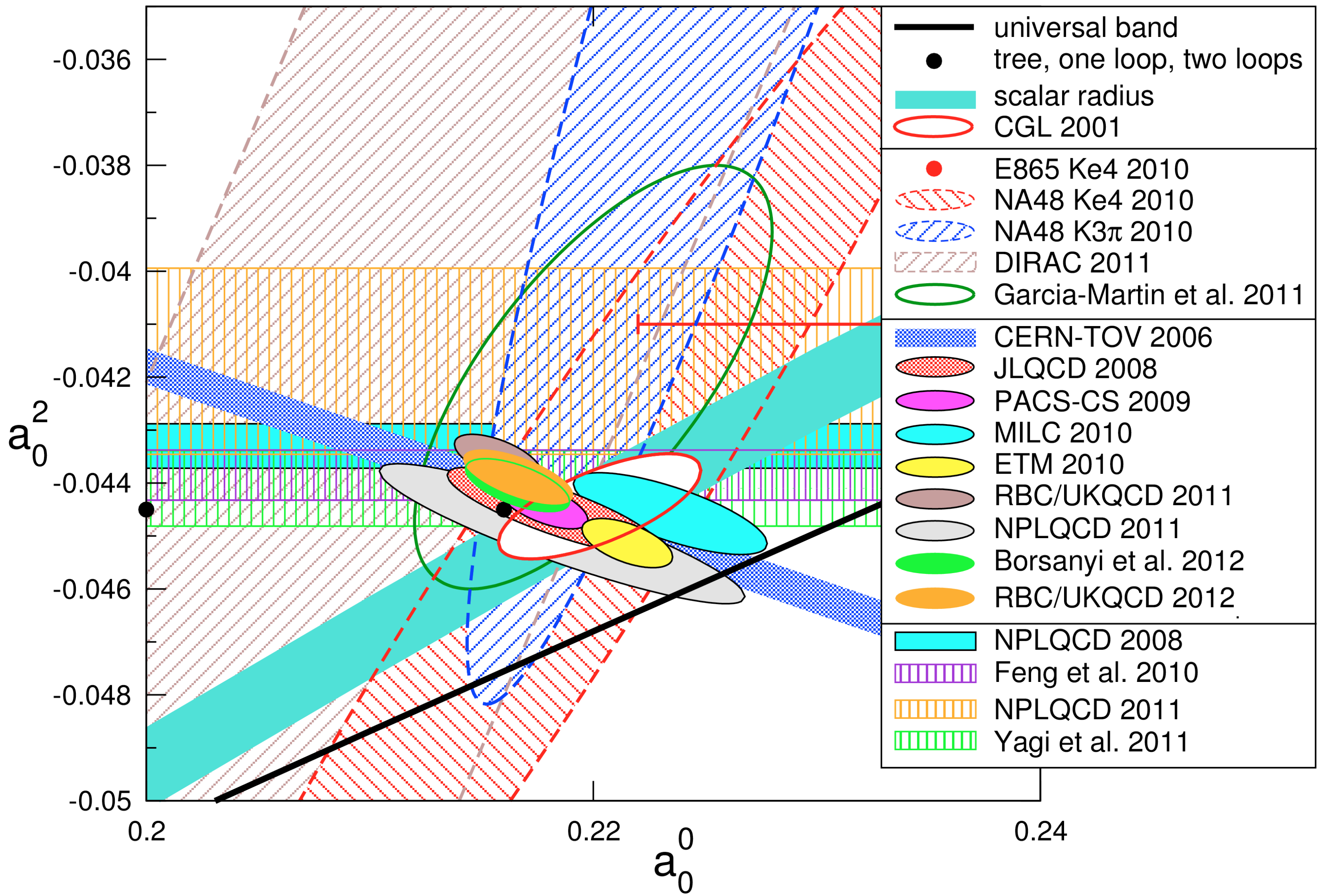
$\pi\pi$ scattering

- I=0 and 2 scattering length
  - corresponding to the cross section.

  - Derivative coupling gives the leading terms of order m_π^2
  - Known to NNLO in χPT; needs the LECs

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left\{1+\frac{9M_\pi^2}{32\pi^2F_\pi^2}\ln\frac{\lambda_{a_0^0}^2}{M_\pi^2}\right\},$$

$$a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left\{1-\frac{M_\pi^2}{32\pi^2F_\pi^2}\ln\frac{\lambda_{a_0^2}^2}{M_\pi^2}\right\}$$



-0.036
-0.038
-0.04
-0.042
-0.044
-0.046
-0.048
-0.05
a_0^2
0.2
0.22
0.24
a_0^0
universal band
tree, one loop, two loops
scalar radius
CGL 2001
E865 Ke4 2010
NA48 Ke4 2010
NA48 K3π 2010
DIRAC 2011
Garcia-Martin et al. 2011
CERN-TOV 2006
JLQCD 2008
PACS-CS 2009
MILC 2010
ETM 2010
RBC/UKQCD 2011
NPLQCD 2011
Borsanyi et al. 2012
RBC/UKQCD 2012
NPLQCD 2008
Feng et al. 2010
NPLQCD 2011
Yagi et al. 2011

Quark mass ratio

- At NLO, the quark mass ratio is given as

$$\frac{m_K^2}{m_\pi^2} = \frac{m_s + m_{ud}}{2m_{ud}} \left[1 + \frac{1}{2} \frac{m_\pi^2}{(4\pi f)^2} \ln \frac{m_\pi^2}{\mu^2} - \frac{1}{2} \frac{m_\eta^2}{(4\pi f)^2} \ln \frac{m_\eta^2}{\mu^2} - \frac{8(m_K^2 - m_\pi^2)}{f^2} (2L_8 - L_5) \right]$$

  - Assumes that the isospin breaking $m_u \neq m_d$ is negligible.
  - Requires the knowledge of the NLO LEC $2L_8$-L_5.
  - Results in m_s/m_{ud}=22~30 (PDG 2010); large uncertainty due to the unknown LEC.

  - Comparison with the exp number gives LECs. But the predictive power is lost.
  - Instead, lattice calculation can be used to fix LECs.

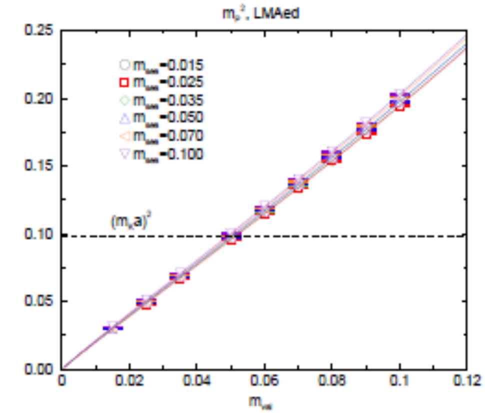
Chiral extrapolation

- Lattice simulation is harder for lighter sea quarks.
  - Computational cost grows as m_q^{-n} (n~2).
  - Finite volume effect becomes more important ~$\exp(-m_\pi L)$

- Practical calculation often involves the *chiral extrapolation*. At the leading order, it is very simple:
  1. Fit the pseudo-scalar mass with $m_\pi^2 = B_0(m_u + m_d) + O(m_q^2)$
  2. Input the physical pion mass $m_{\pi 0}$=135 MeV to obtain $m_{ud}=(m_u+m_d)/2$. (Forget about the isospin breaking for the moment.)
  3. Renormalize it to the continuum scheme to obtain the value in MSbar

NLO example

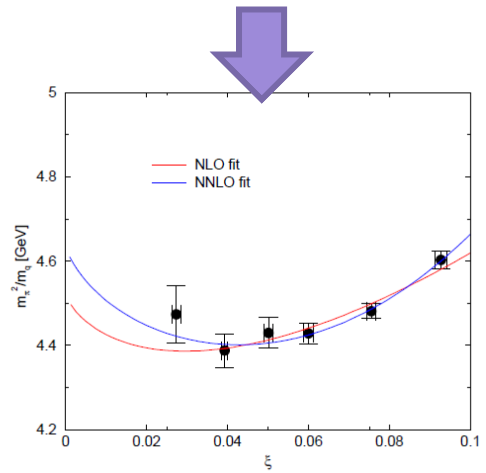
JLQCD (2007)
dynamical overlap (N_f=2)



Chiral expansion

$$m_\pi^2 = 2B_0 m_q \left[1 + \frac{1}{2} \frac{m_\pi^2}{(4\pi f_\pi)^2} \ln \frac{m_\pi^2}{\mu^2} + c_3 \frac{m_\pi^2}{(4\pi f_\pi)^2} + \text{NNLO} \right]$$

- LO (linearity) looks very good, but if you look more carefully NLO is visible.



- m_π^2/m_q not constant.
- Chiral log term has a definite coefficient = curvature fixed.
- Analytic term has an unknown constant, to be fitted with lattice data = linear slope

Chiral symmetry is important!

- So, the realization of chiral symmetry is of crucial importance for lattice calculations.

- Wilson:
  - chiral symmetry is lost: Need modified χPT

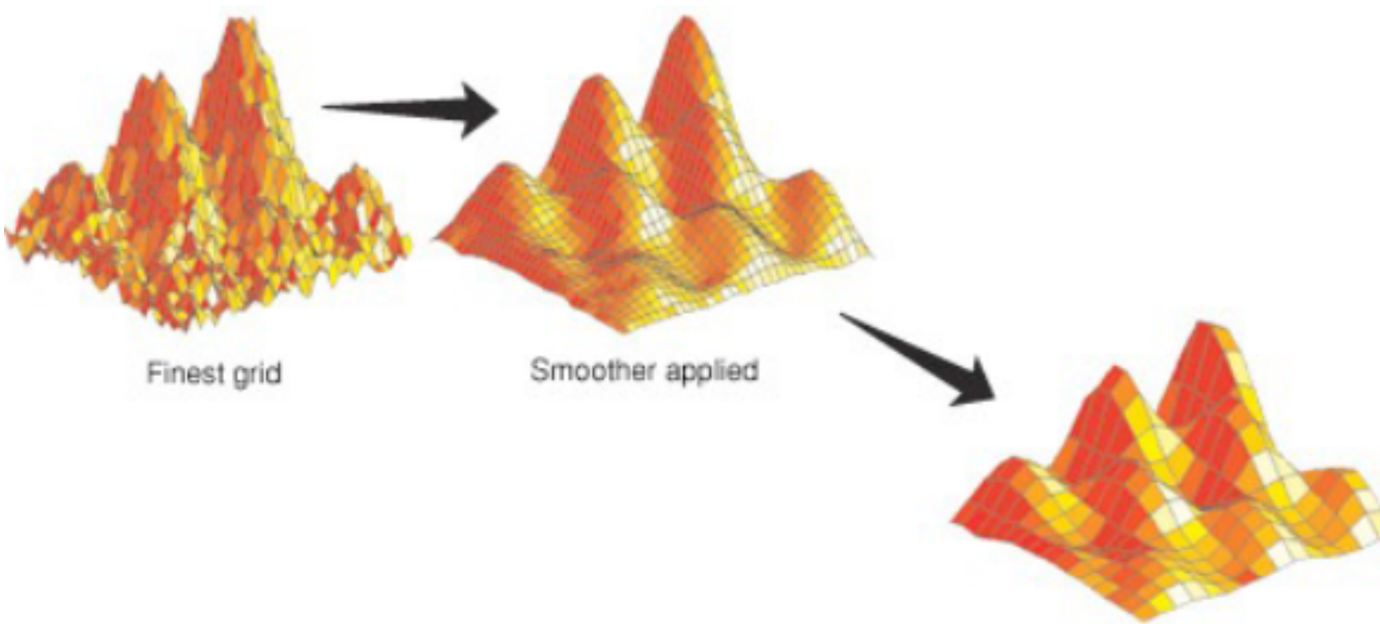
- Staggered:
  - extra tastes are involved: Need modified χPT

- Domain-wall, Overlap:
  - No need, but costly.

2 Lattice gauge theory

2.4 Controlling the systematic effects

Discretization



Need fine grids to approximate the continuum.
What is the necessary resolution?

Multi-scale problem

- Players of QCD span between 3 MeV and 5 GeV
  - Not feasible (for now) to treat at once.

    (Nuclear physics is not considered here.)

$\sim 1/m_q^{(2\text{-}3)}$ **LQCD** $\sim 1/a^7$

$$m_{ud} \ll m_s < \Lambda_{QCD} \ll m_c < m_b \ll m_W$$

m_{ud}	m_s	Λ_{QCD}	m_c	m_b	m_W
~3 MeV	~ 100 MeV	~ 300 MeV	~1.5 GeV	~4.5 GeV	80 GeV

- Plus, arbitrary momentum scale appear in QFT.
  - Physically irrelevant scale can be integrated out; its effects are encoded in the coupling constant = Renormalization Group.

Multi-scale problem

- Players of QCD span between 3 MeV and 5 GeV
  - Not feasible (for now) to treat at once.

    (Nuclear physics is not considered here.)

$\sim 1/m_q^{(2\text{-}3)}$ **LQCD** $\sim 1/a^7$

$$m_{ud} \ll m_s < \Lambda_{QCD} \ll m_c < m_b \ll m_W$$

m_{ud}	m_s	Λ_{QCD}	m_c	m_b	m_W
~3 MeV	~ 100 MeV	~ 300 MeV	~1.5 GeV	~4.5 GeV	80 GeV

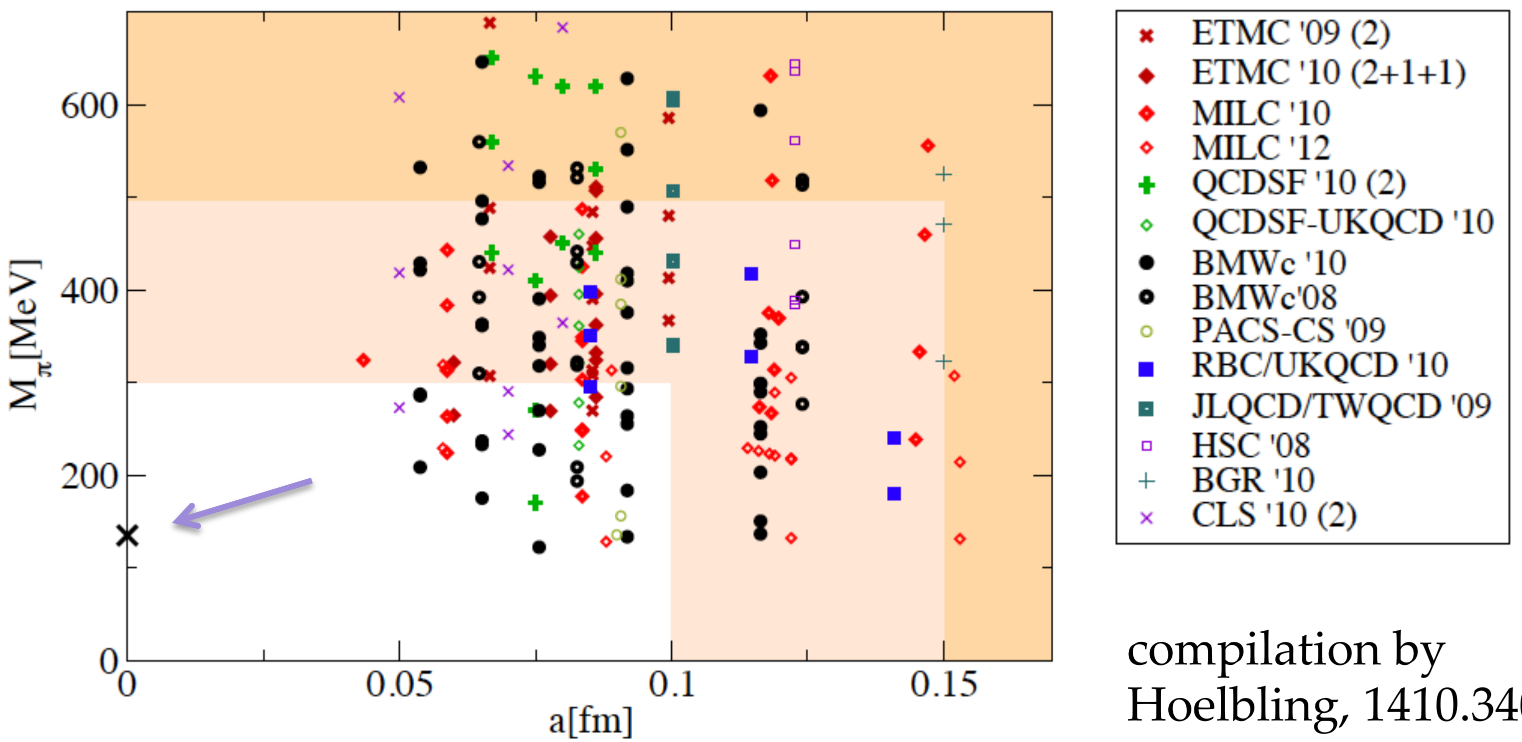
- Two directions (or both)

  ← Going to the physical up/down quark masses

  → Fine lattices to directly treat charm (or even bottom)

Simulation parameters

- Approaching the continuum/physical limit



compilation by
Hoelbling, 1410.3403

Discretization effect

- Understood using an effective field theory (Symanzik).

$$\mathcal{L}_{\text{Sym}} = \mathcal{L}^{(4)} + a\mathcal{L}^{(5)} + a^2\mathcal{L}^{(6)} + \cdots$$

  - $L^{(4)}$ is the same as the continuum QCD.
  - $L^{(5)}$, $L^{(6)}$, … represent the discretization effects. All possible operators of that mass dimension may appear.
  - All "possible" operators allowed by the lattice symmetry.

$$\mathcal{L}^{(5)} \ni \bar{\psi}D_\mu^2\psi,\ \ \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$$

violates chiral symmetry; allowed for Wilson, not for DW/OV

$$\not\ni \bar{\psi}\gamma_5 D_\mu^2\psi$$

violates parity; not allowed for lattice actions respecting parity

Discretization effect

- Understood using an effective field theory (Symanzik).

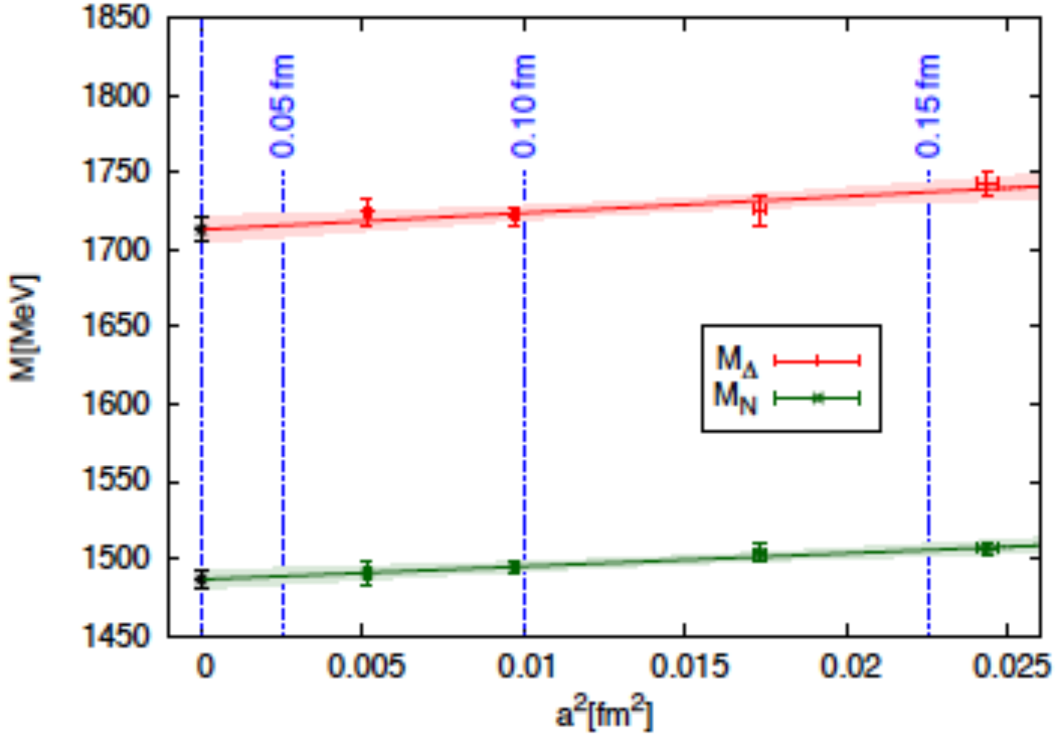
$$\mathcal{L}_{\mathrm{Sym}} = \mathcal{L}^{(4)} + a\mathcal{L}^{(5)} + a^2\mathcal{L}^{(6)} + \cdots$$

  - $L^{(4)}$ is the same as the continuum QCD.
  - $L^{(5)}$, $L^{(6)}$, … represent the discretization effects. All possible operators of that mass dimension may appear.
  - All "possible" operators allowed by the lattice symmetry.

  - Typically, the O(a) error is eliminated; the leading error is O(a^2).

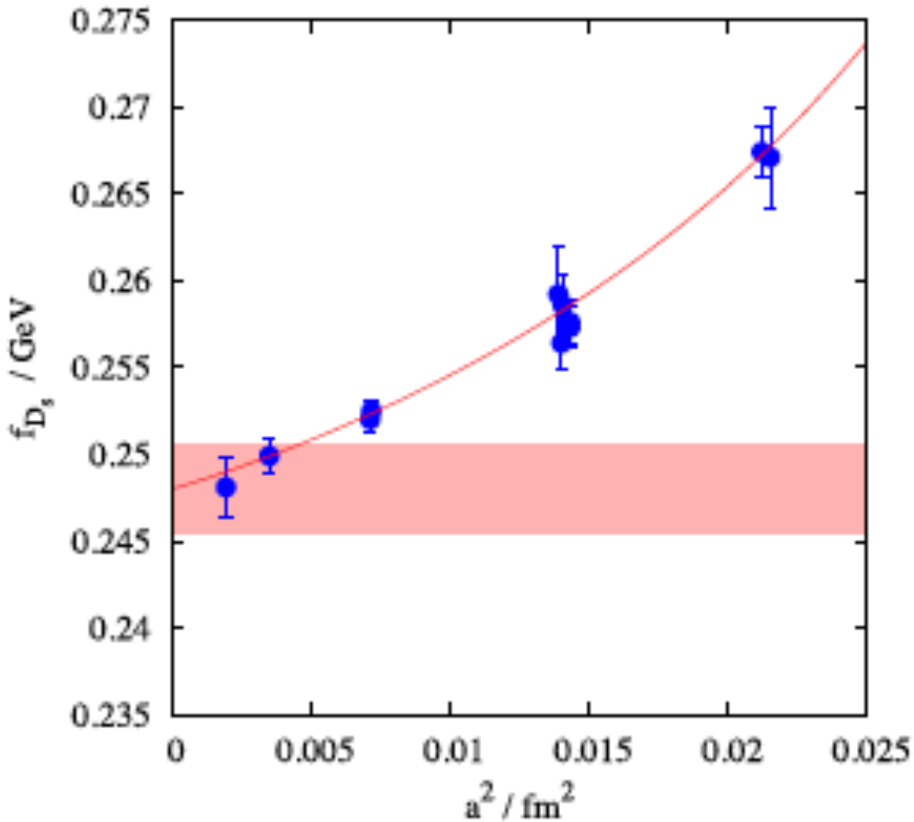
Continuum limit

BMW (2011)



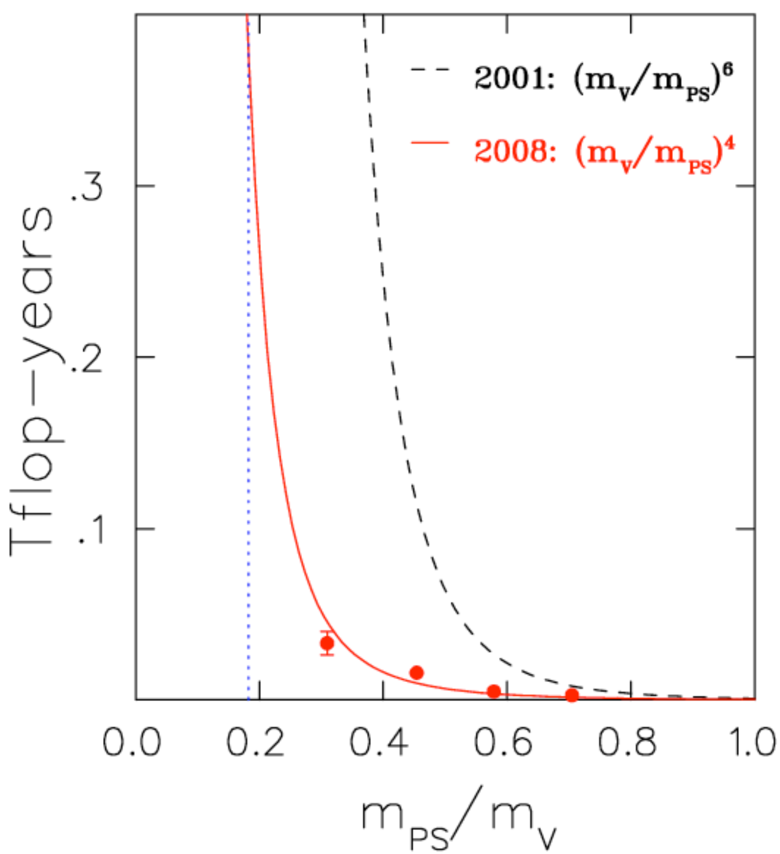
Continuum limit

HPQCD (2010)



Light quark masses

ETMC (2008)

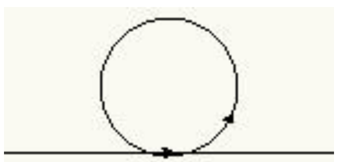


- Computational cost grows for lighter light quarks
  - $1/m_q$ for inversion
  - $1/m_q$ for integration
  - $1/m_q$ for autocorrelation
- Improved over years
  - new algorithms
  - new machines

Now feasible to simulate at physical up/down quark masses

Light quark masses

- Important because the quark mass dependence could be non-trivial.
  - nearly massless pions may introduce non-analytic behavior.



$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} = \frac{1}{(4\pi)^2}\left[\Lambda^2 + m^2 \ln\frac{m^2}{\Lambda^2+m^2}\right]$$ Cutoff regularization

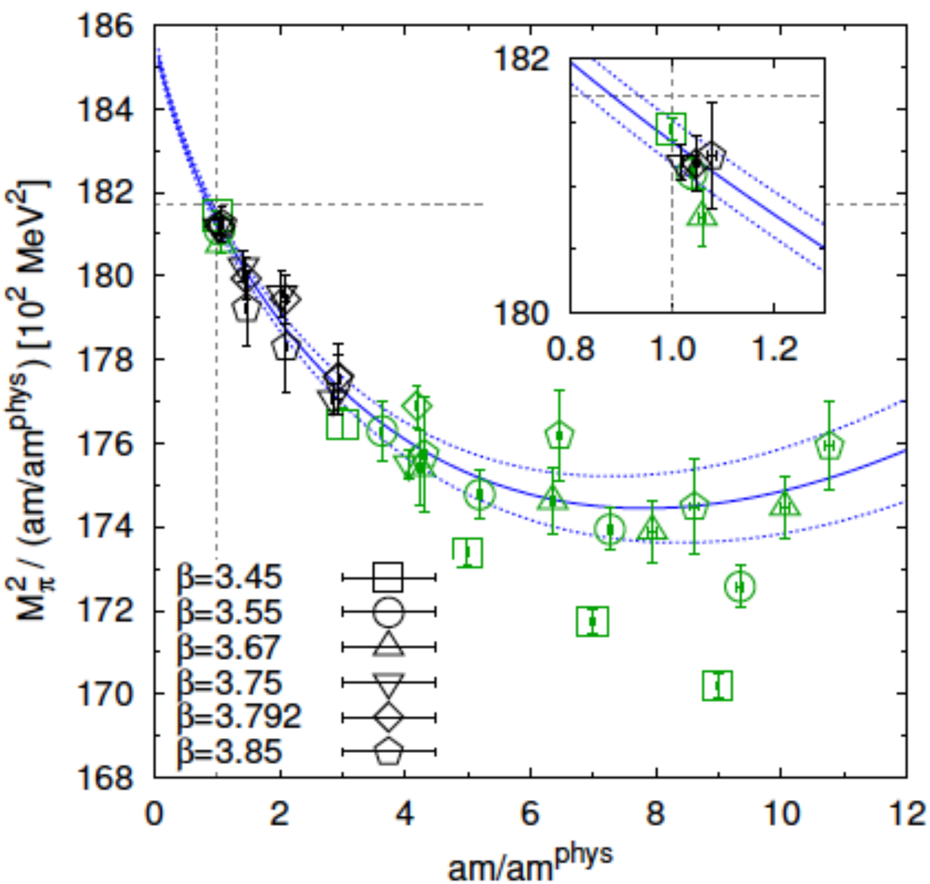
$$= \frac{m^2}{(4\pi)^2}\left(\frac{2}{\varepsilon}+\gamma-\ln 4\pi+\ln\frac{m^2}{\mu^2}-1\right)$$ Dimensional reg

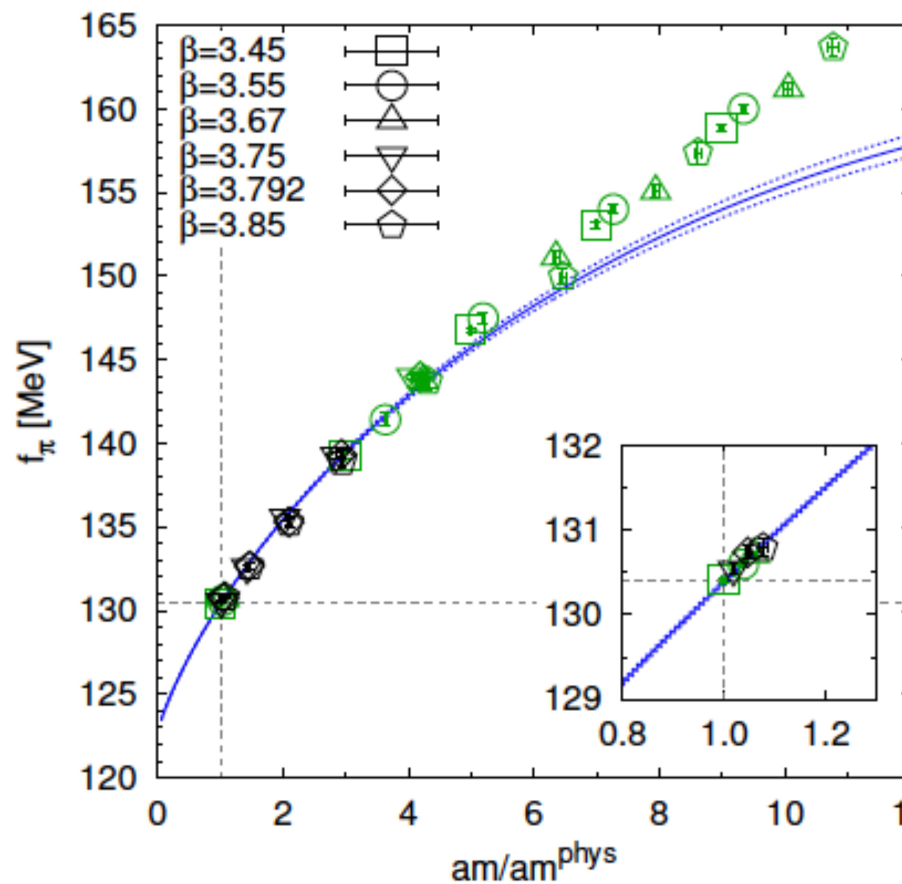
  - $m^2 \ln m^2$ is called the chiral log.
  - After subtracting the UV divergence

$$m_\pi^2 = 2B_0 m_q\left[1+\frac{1}{2}\frac{m_\pi^2}{(4\pi f)^2}\ln\frac{m_\pi^2}{\mu^2}+(\mathrm{const})\times\frac{m_\pi^2}{(4\pi f)^2}+O(m_\pi^4)\right]$$

Quark mass dependence

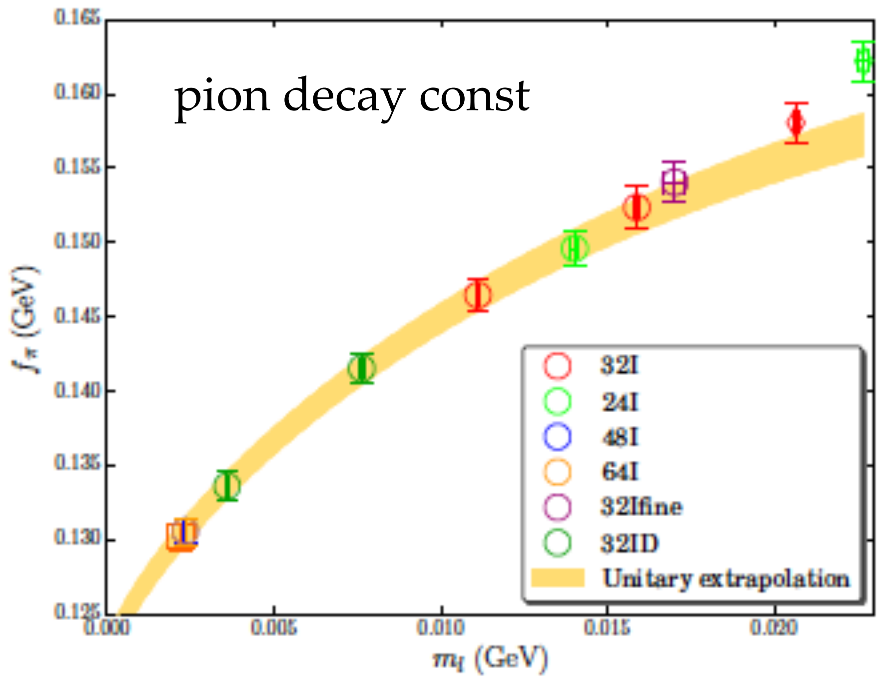
BMW (2014)

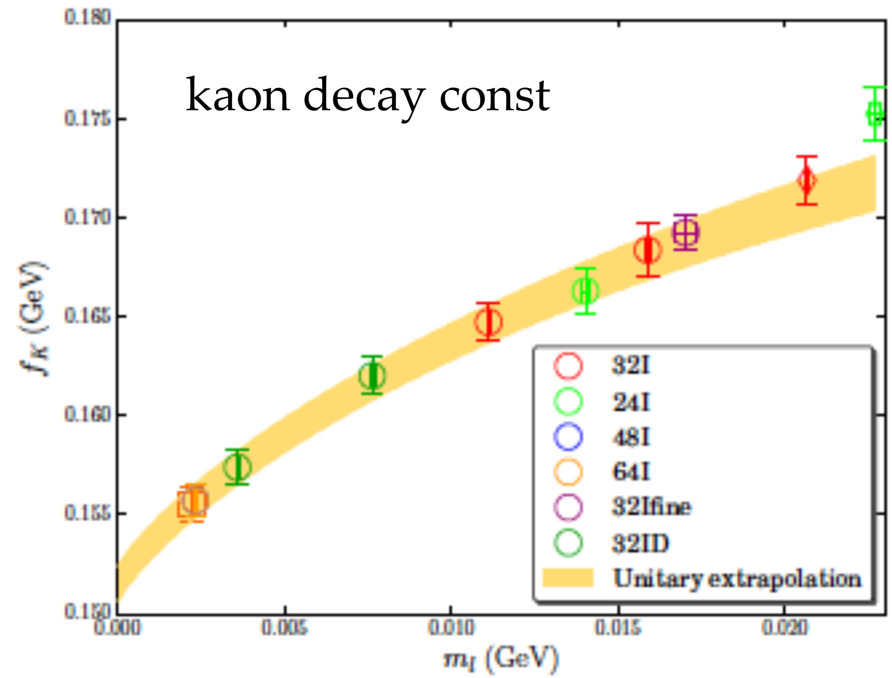




Quark mass dependence

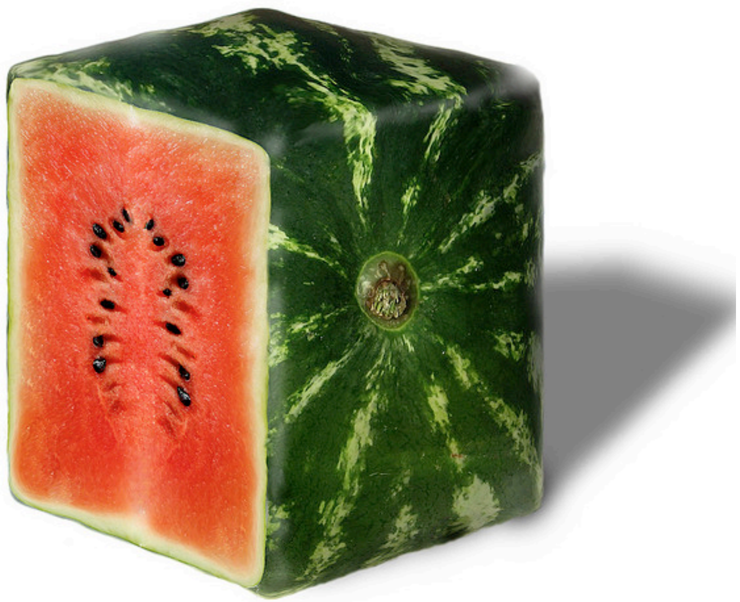
RBC/UKQCD, arXiv:1411.7017





Extrapolation to the physical point is non-trivial.

Finite volume

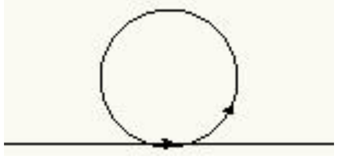


- Lattice needs to be larger than the size of nucleon
  - c.f. proton charge radius ~ 0.9 fm.
- How large? Associated error should be carefully studied.
  - Biggest effect would be from pions.

Finite volume effect

- Obvious constraint is from the QCD scale $1/\Lambda_{QCD}$. But it is smaller than the length scale of pion $1/m_\pi$.

- Can be understood again using chiral effective theory.

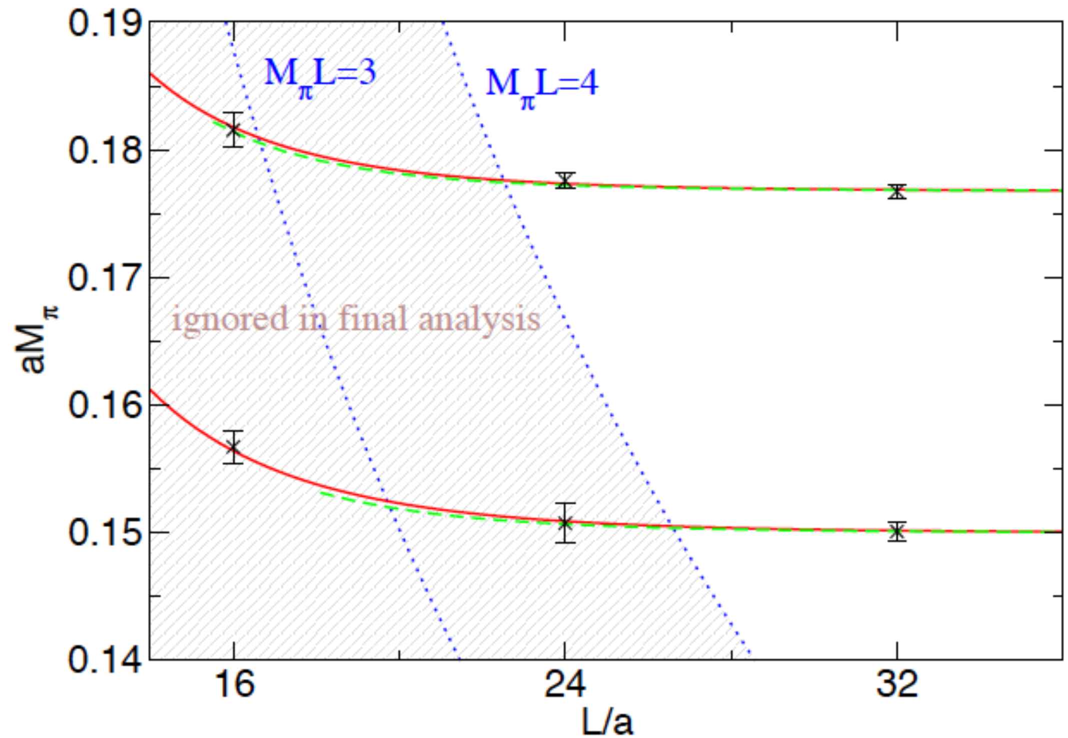


$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \to \sum_k \frac{1}{k^2+m^2}$$

  - The effect is like exp(−mL).
  - Suppressed to sub-% level at mL~4.

Infinite volume limit

Finite volume: BMW (2011)



Heavy quark

- $m_c \sim 1.5$ GeV, $m_b \sim 4.5$ GeV: not small compared to the (currently available) lattice cutoff $1/a$.
  - Compton wavelength is smaller than the lattice spacing.
  - Significant discretization effects.

- Still, the relevant scale should be lower for low-energy dynamics. Some effective theory may be introduced.
  - $m_Q \gg \Lambda_{QCD}$,: Heavy quark effective theory (for heavy-light)
  - $m_Q \gg m_Q\alpha_s$: Non-relativistic QCD (for heavy-heavy)

  - No wonderful trick for energetic processes.

Charm and bottom

- Heavy-light (D, B)

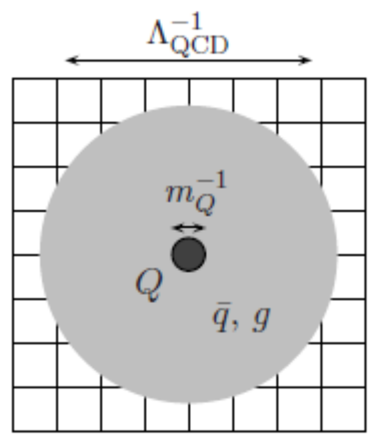
  $$m_H = m_Q + E_{Q\bar{q}}$$

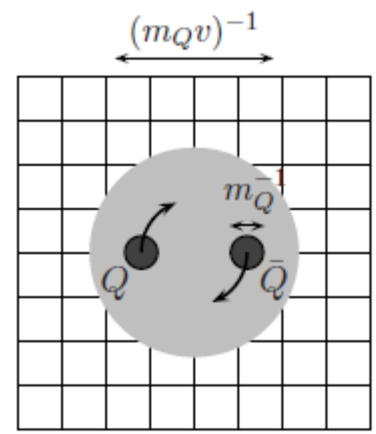
  - E_{Qq} denotes a binding energy.
  - Simply calculate the meson mass; tune m_Q until m_H reproduces the experimental value.
  - Calculate E_{Qq}, whose m_Q dependence is subleading. Then, m_H-E_{Qq} gives m_Q. (Heavy Quark Symmetry)

- Heavy-heavy (J/ψ, Υ)

  $$m_H = m_Q + m_{\bar{Q}} + E_{Q\bar{Q}}$$

  - E_{QQ} denotes a binding energy.
  - Binding energy crucially depends on m_Q.





Heavy Quark Effective Theory (HQET)

- Write the momentum of heavy quark as $p = m_Q v + k$
  - v : four-velocity of the heavy quark.
  - k: residual momentum
- Heavy quark mass limit:
  - propagator

$$i\frac{\not{p}+m_Q}{p^2-m_Q^2+i\varepsilon} = i\frac{m_Q\not{v}+m_Q+\not{k}}{2m_Q v\cdot k+k^2+i\varepsilon} \rightarrow i\frac{1+\not{v}}{2}\frac{1}{v\cdot k+i\varepsilon}$$

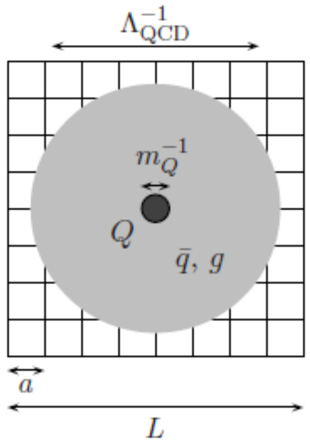
  - Lagrangian

$$L_Q = \bar{Q}_v(iv\cdot D)Q_v; \quad Q(x) = e^{-im_Q v\cdot x}Q_v(x)$$

Georgi (1990), Eichten-Hill (1990)

  - Heavy quark mass drops out from the dynamics = Heavy Quark Symmetry

Isgur-Wise (1989)

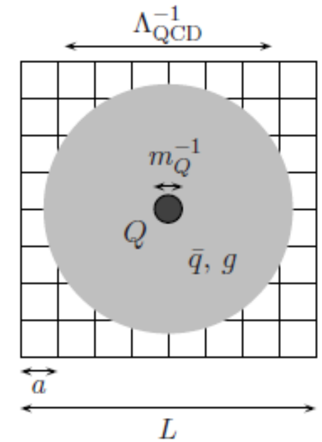


HQET on the lattice

- Discretize the HQET lagrangian
  - Assuming $v^\mu=(1,0)$: rest frame of the heavy quark

$$S_Q = \sum_x Q^+(x)[Q(x) - U_4^+(x-\hat{4})Q(x-\hat{4})]$$

  - Heavy quark propagator becomes a static color source.
  - Heavy-light meson mass: $m_H = m_Q + E_{Q\bar{q}}$
    Calculate E_{Qq}, then, m_H-E_{Qq} gives m_Q up to Λ_{QCD}/m_Q corrections.



Limitation of effective theory

- Obviously, HQET (at LO) ignores the $1/m_Q$ effects.
  - Higher order terms can be included. The leading corrections:

$$H = -\frac{D^2}{2m_Q} - \frac{\sigma \cdot B}{2m_Q}$$

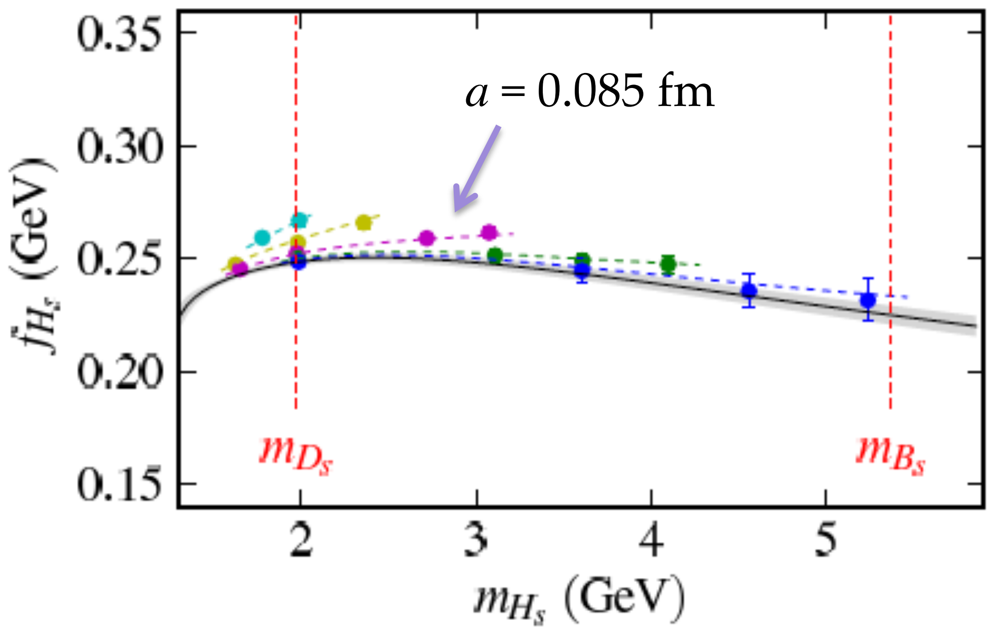
  - The coefficients of terms are constrained by the Lorentz invariance, thus giving $1/2m_Q$.
  - But, in the quantum theory they are renormalized differently, since the Lorentz invariance is violated by the choice of the reference frame v^μ.

$$H = -\frac{D^2}{2(Z_m m_Q)} - c_B \frac{\sigma \cdot B}{2(Z_m m_Q)}$$

  - The coefficients (Z_m and c_B) must be calculated (non-)perturbatively.
  - The same complication arises at every order of the expansion.

Heavy quark (conventional)

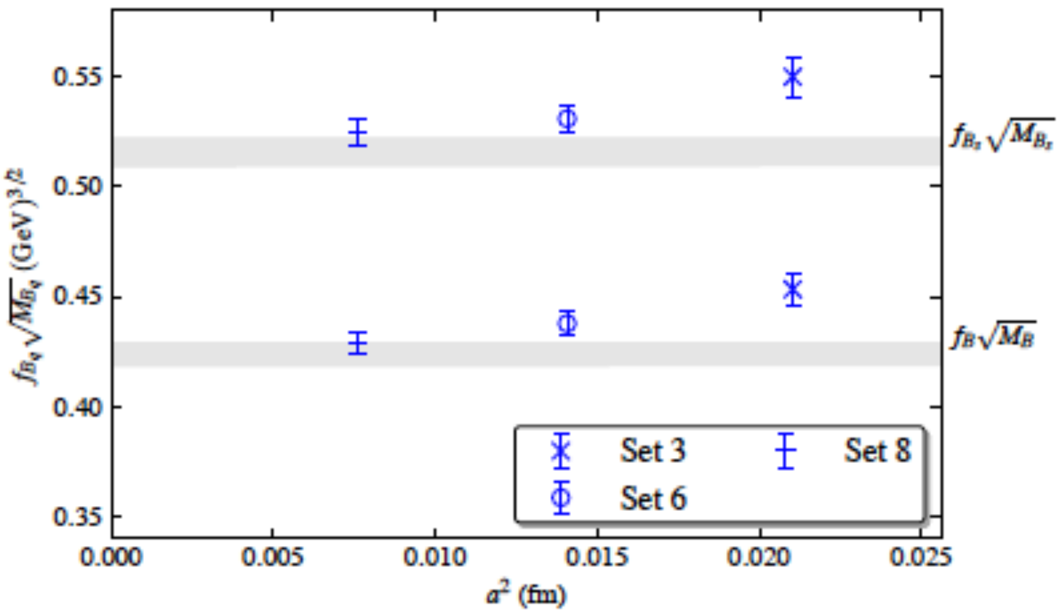
Heavy-light meson decay constant: HPQCD (2011)



Lattice spacing between a = 0.145 fm and 0.044 fm.

Heavy quark (NRQCD)

HPQCD (2013)

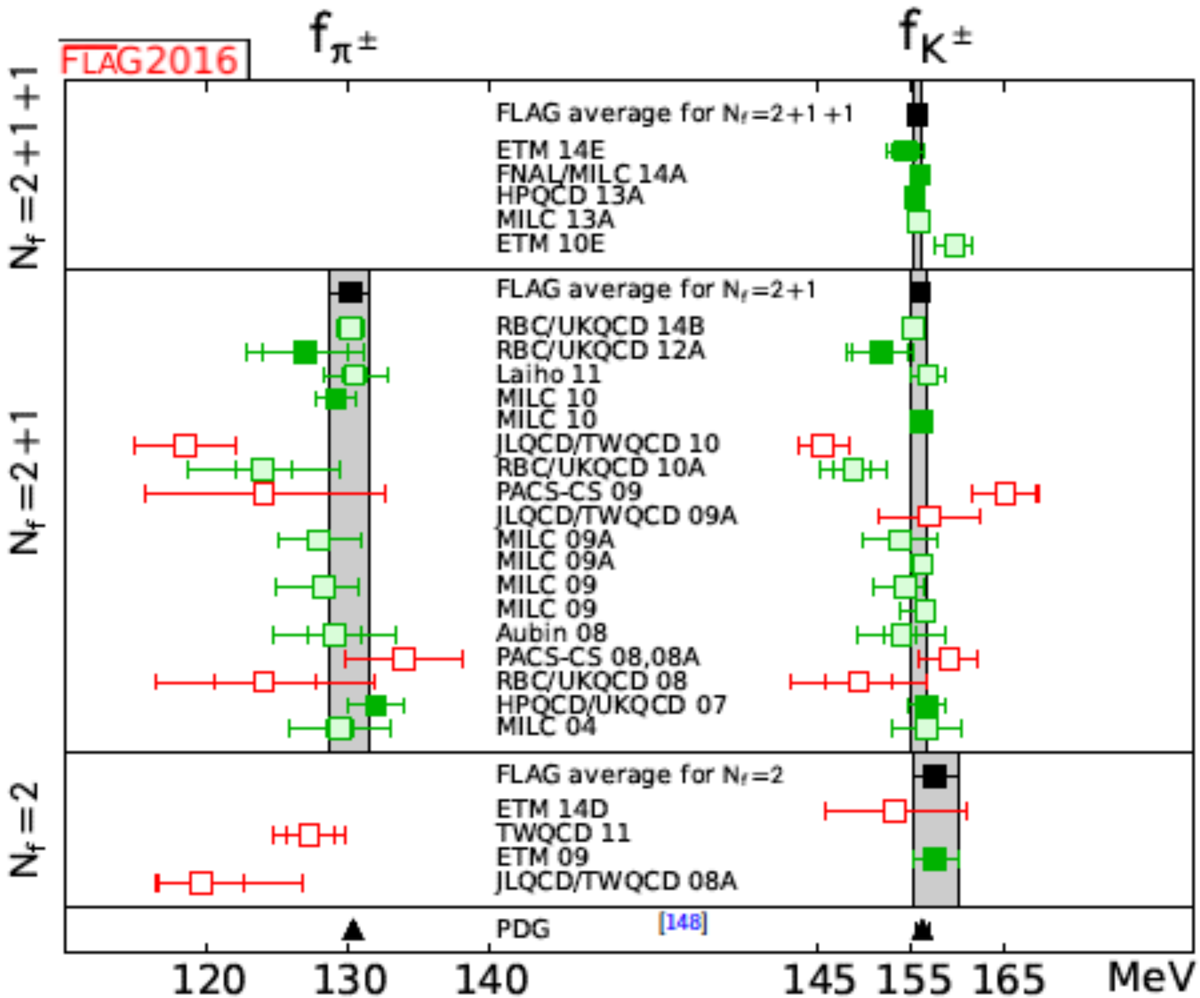


All errors taken into account

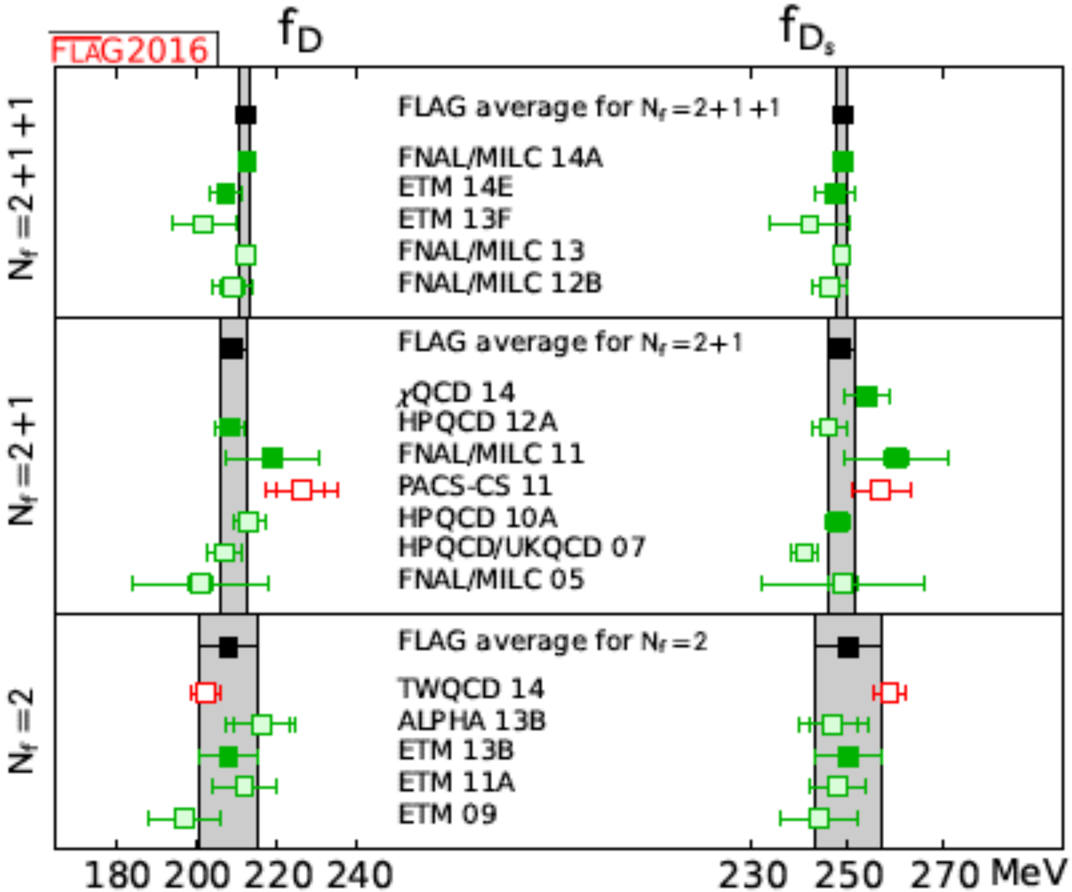
FLAG (2016)

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	m_{ud}	m_s
ALPHA 12	[12]	A	○	★	★	★	a, b		102(3)(1)
Dürr 11[‡]	[132]	A	○	★	○	–	–	3.52(10)(9)	97.0(2.6)(2.5)
ETM 10B	[11]	A	○	★	○	★	c	3.6(1)(2)	95(2)(6)
JLQCD/TWQCD 08A	[138]	A	○	■	■	★	–	4.452(81)(38)$\left(^{+0}_{-227}\right)$	–
RBC 07[†]	[105]	A	■	■	★	★	–	4.25(23)(26)	119.5(5.6)(7.4)
ETM 07	[133]	A	○	■	○	★	–	3.85(12)(40)	105(3)(9)
QCDSF/ UKQCD 06	[139]	A	■	★	■	★	–	4.08(23)(19)(23)	111(6)(4)(6)
SPQcdR 05	[140]	A	■	○	○	★	–	4.3(4)$\left(^{+1.1}_{-0.0}\right)$	101(8)$\left(^{+25}_{-0}\right)$
ALPHA 05	[135]	A	■	○	★	★	a		97(4)(18)[§]
QCDSF/ UKQCD 04	[137]	A	■	★	■	★	–	4.7(2)(3)	119(5)(8)
JLQCD 02	[141]	A	■	■	○	■	–	3.223$\left(^{+46}_{-69}\right)$	84.5$\left(^{+12.0}_{-1.7}\right)$
CP-PACS 01	[134]	A	■	■	★	■	–	3.45(10)$\left(^{+11}_{-18}\right)$	89(2)$\left(^{+2}_{-6}\right)$[⋆]

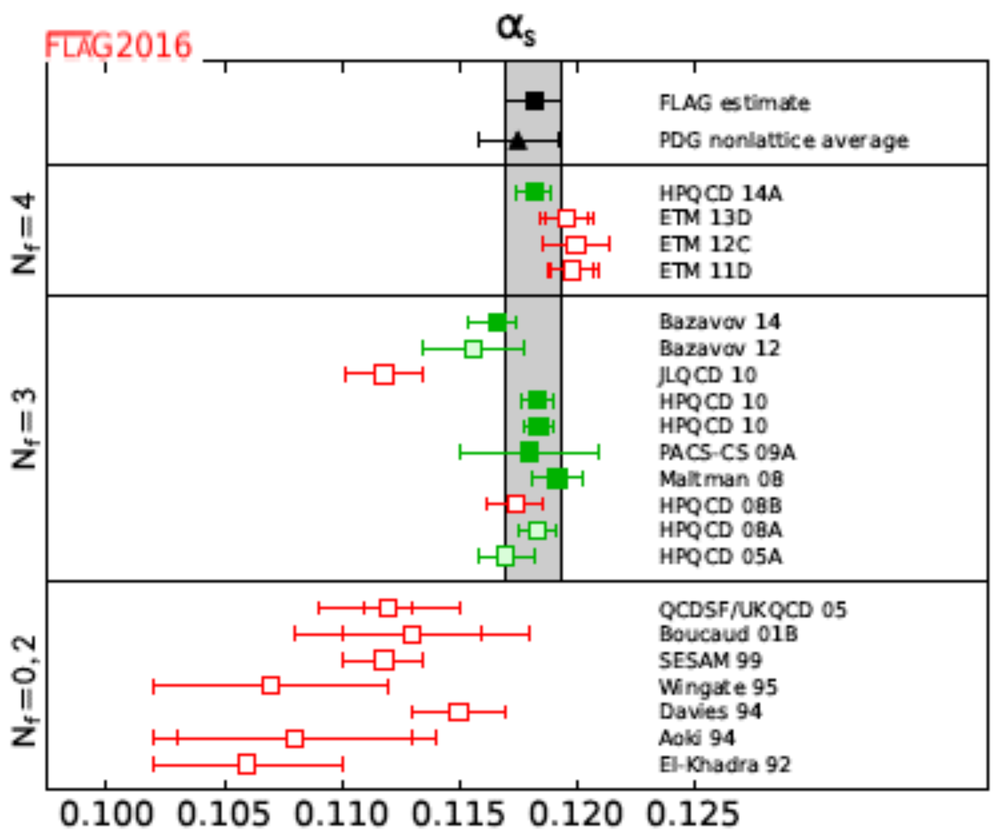
Leptonic decay constants



Leptonic decay constants



Strong coupling constant

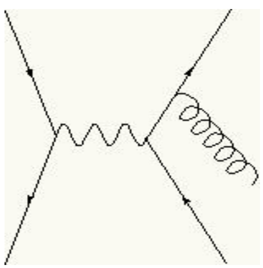


3. Application to particle phenomenology

3.1 Use and limitation of perturbation theory

Perturbation theory?

- One can treat only plane wave of quark/gluon field as the initial/final states, and not hadrons. What can we calculate, then?
- For instance, the sum of all possible final states.



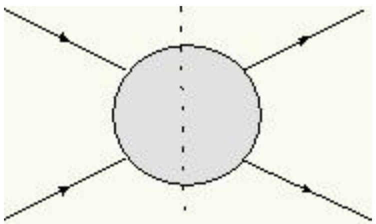
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$= 3\sum_q Q_q^2 \left\{1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right\}$$

  - They must be quarks, initially. They then hadronize, and the cross section must be the same. = Quark-hadron duality (which is an assumption).

Optical theorem

- Unitarity of scattering amplitude



$$S = I + iT$$

$$I = S^\dagger S = (I - iT^\dagger)(I + iT) = I + i(T - T^\dagger) + T^\dagger T$$

$$\Rightarrow T^\dagger T = 2\,\mathrm{Im}\,T$$

cross section (sum of final states)

imaginary part of $e^+e^- \rightarrow e^+e^-$

  - No hadrons in the initial/final states. Perturbation theory can be applied.
  - Is it true? The internal states are hadrons.

Quark-hadron duality

[assumption] cross section for hadronic final states can be calculated using quarks.

- The key is the sum over final states… a smearing

Poggio, Quinn, Weinberg, PRD13, 1958 (1976)

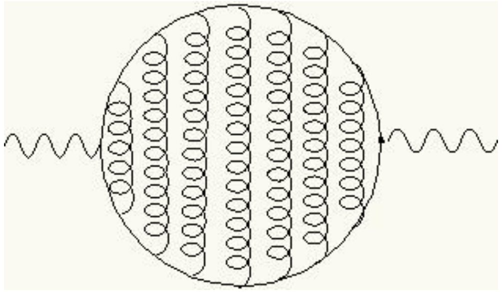
$$\bar{R}(s,\Delta) \equiv \frac{\Delta}{\pi}\int_0^\infty ds' \frac{R(s')}{(s-s')^2+\Delta^2}$$

$$= \frac{1}{2\pi i}\int_0^\infty ds' R(s')\left(\frac{1}{s-s'+i\Delta} - \frac{1}{s-s'-i\Delta}\right)$$

$$= \frac{1}{2i}\left[\Pi(s+i\Delta) - \Pi(s-i\Delta)\right]$$

may avoid resonances; perturbative expansion is convergent.

higher orders become important near resonances

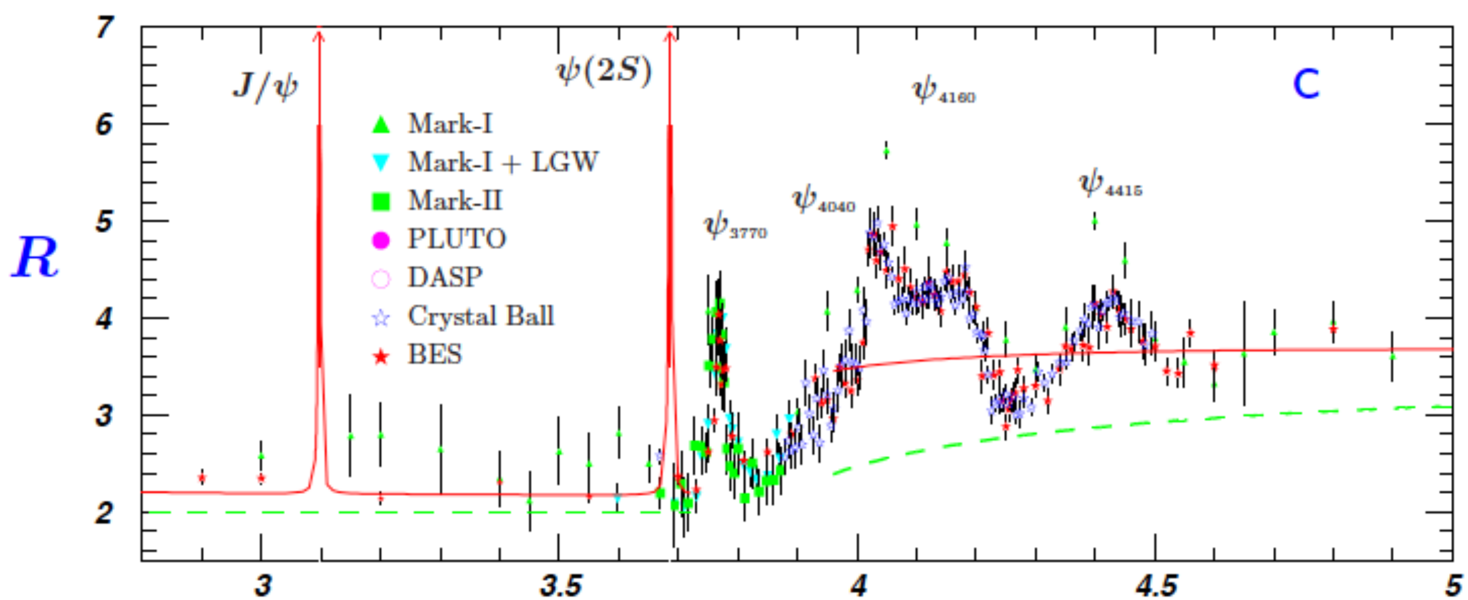


$$\mathrm{Im}\Pi(s) \propto R(s) = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Quark-hadron duality

[assumption] cross section for hadronic final states can be calculated using quarks.

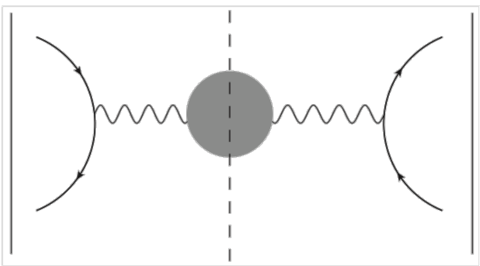
- The key is the sum over final states… a smearing.
- Need sufficient smearing to avoid the resonance effect.

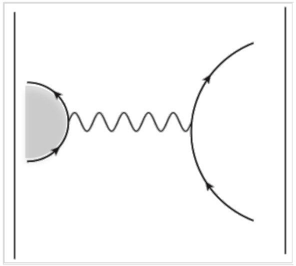


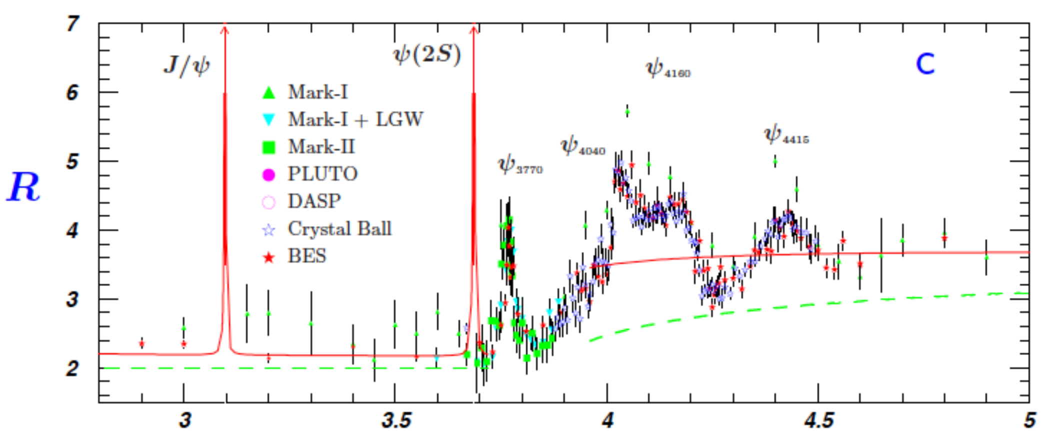
Charmonium correlator

- Theory vs exp, through moments

$$\frac{1}{n!}\left(\frac{\partial}{\partial q^2}\right)^n \left(\Pi(q^2)\right)_{q^2=0} = \frac{1}{12\pi Q_f^2}\int \mathrm{d}s \frac{1}{s^{n+1}} R(s)_{e^+e^-\to \text{ hadron}}$$







Charmonium correlator

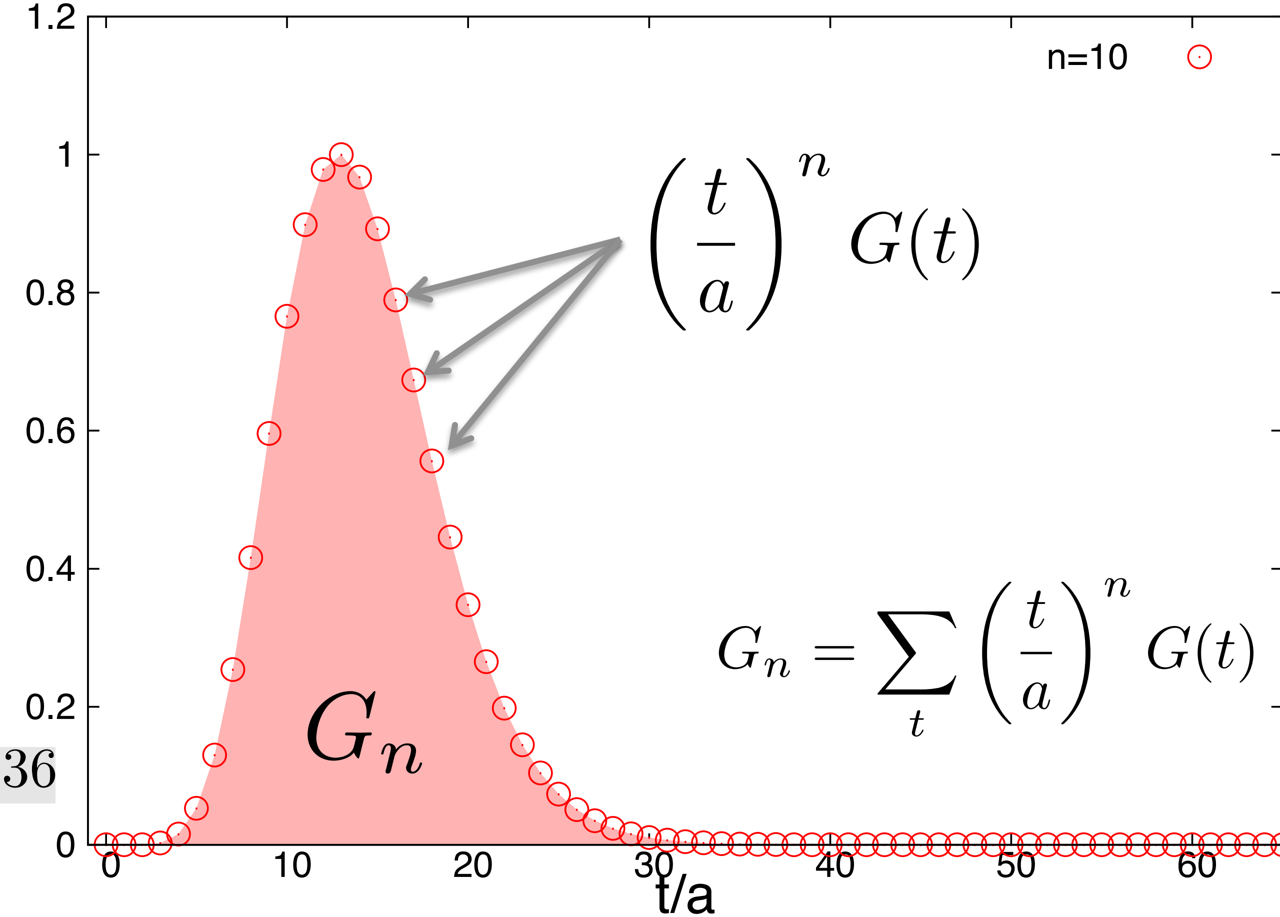
- Moments on the Euclidean lattice

$$i\int \mathrm{d}x \frac{1}{n!}\left(\frac{\partial}{\partial q^2}\right)^n \mathrm{e}^{iqt} \longrightarrow a^4 \sum_x t^{2n}$$

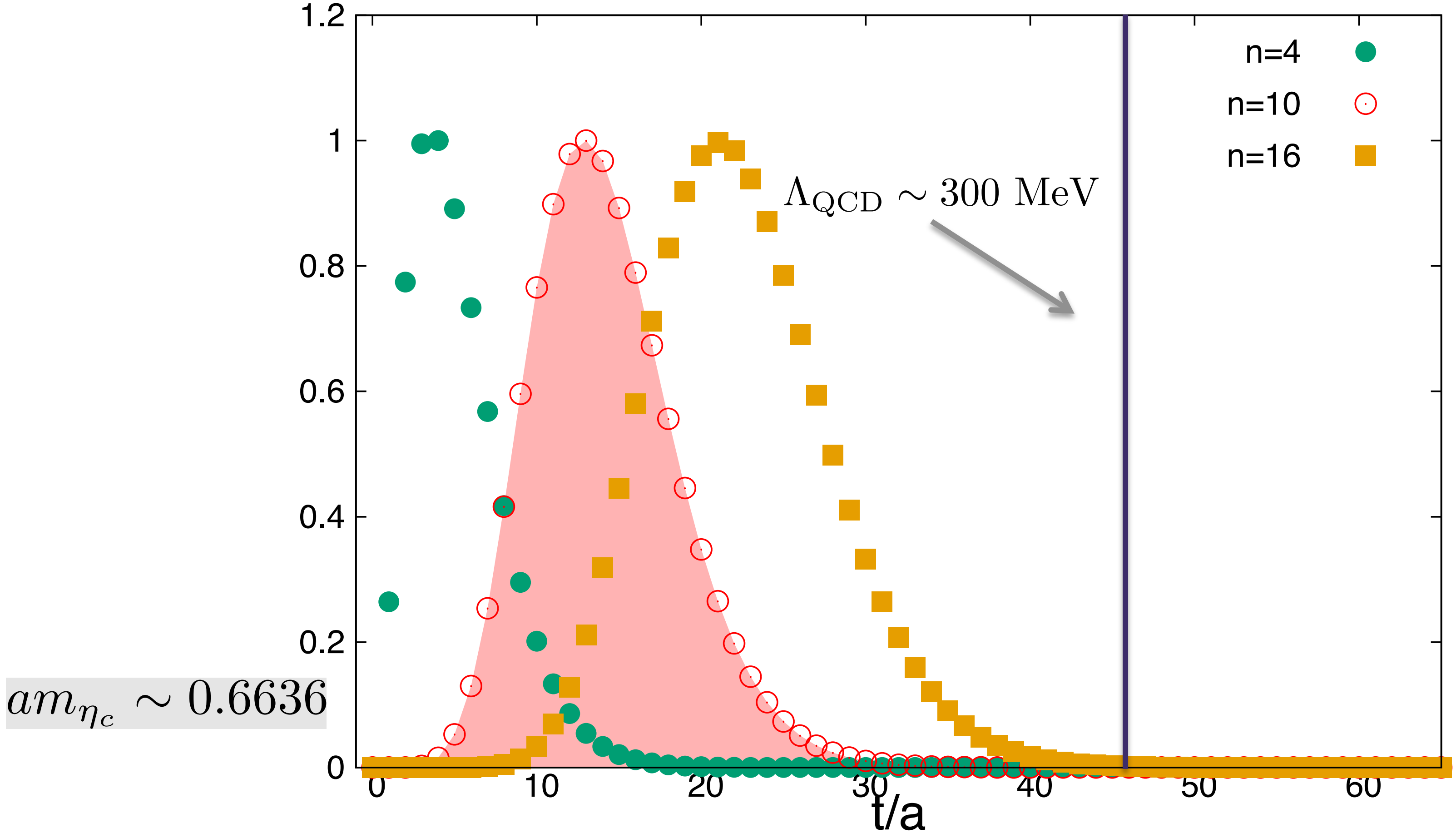
  - Simply constructed from the correlators

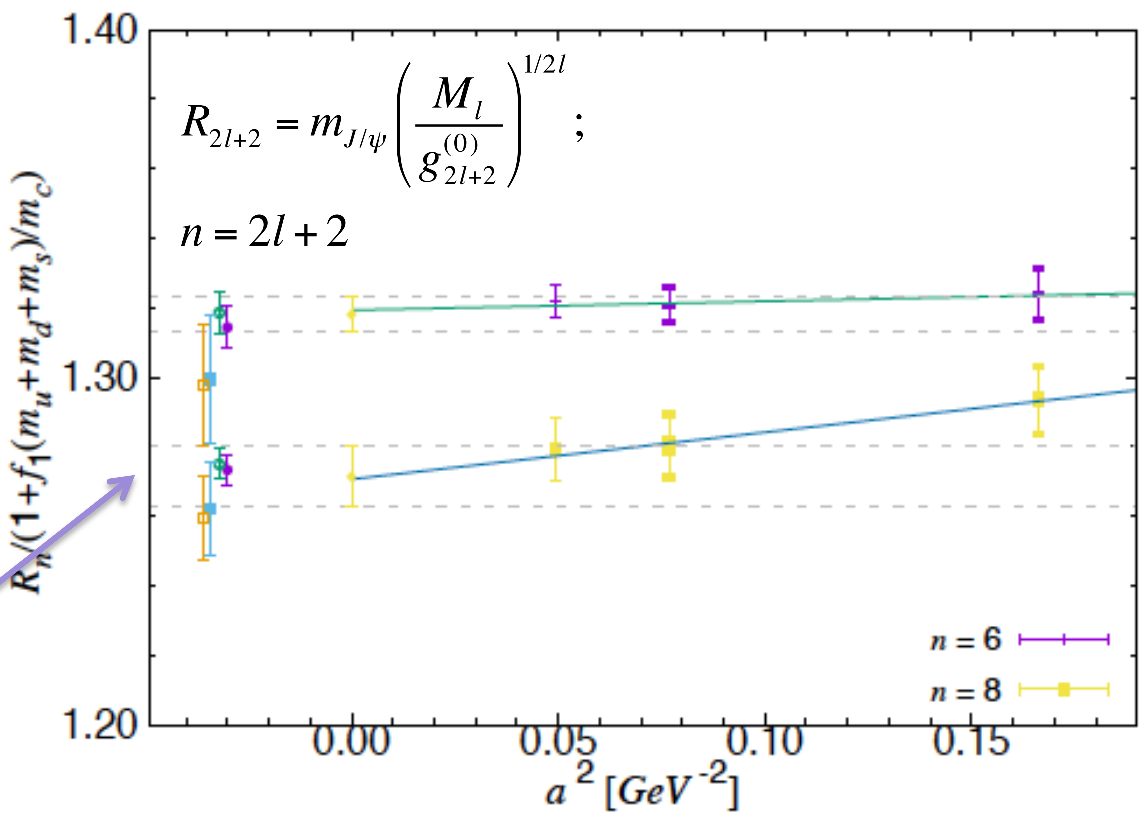
$$G_V(t) = a^6 \sum_x \langle 0 | j_k(x,t) j_k(0,0) | 0 \rangle, \quad G_{V,n} = \sum_t (t/a)^n G_V(t)$$

  - $G_V(t)$ represents a J/ψ correlator, ~exp($-m_{J/\psi}t$), plus its excited states, continuum, etc.

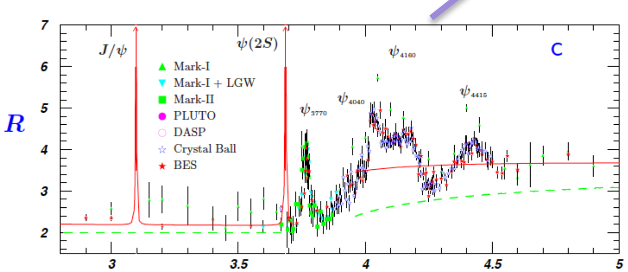


$am_{\eta_c} \sim 0.6636$


1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
t/a
n=4
n=10
n=16
$\Lambda_{\mathrm{QCD}} \sim 300\ \mathrm{MeV}$
$am_{\eta_c} \sim 0.6636$


$R_{2l+2} = m_{J/\psi}\left(\frac{M_l}{g^{(0)}_{2l+2}}\right)^{1/2l}$;
$n = 2l + 2$
$R_n/(1+f_1(m_u+m_d+m_s)/m_c)$
1.40
1.30
1.20
0.00
0.05
0.10
0.15
a^2 [GeV^{-2}]
$n = 6$
$n = 8$

Exp data


R
J/ψ
$\psi(2S)$
Mark-I
Mark-I + LGW
Mark-II
PLUTO
DASP
Crystal Ball
BES
ψ_{3770}
ψ_{4040}
ψ_{4160}
ψ_{4415}
c
2
3
4
5
6
7
3
3.5
4
4.5
5

Charm quark mass

- Method developed by HPQCD/Karlsruhe (2008~)

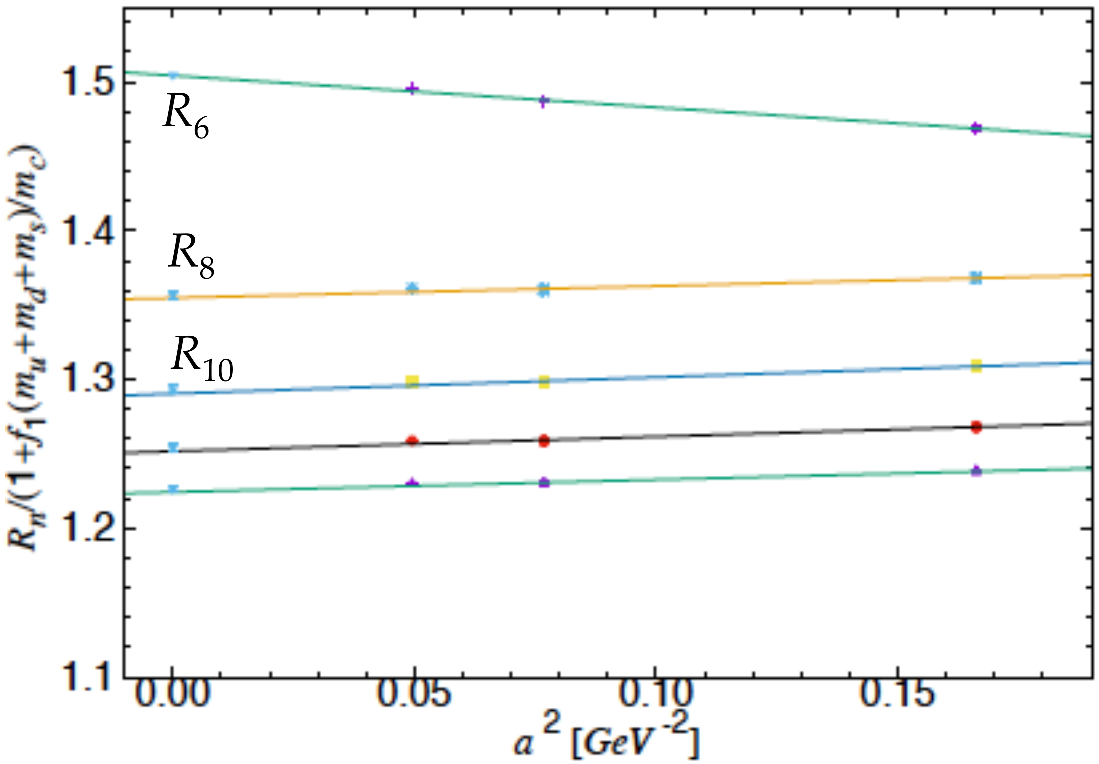
Lattice

Pert to α_s^3

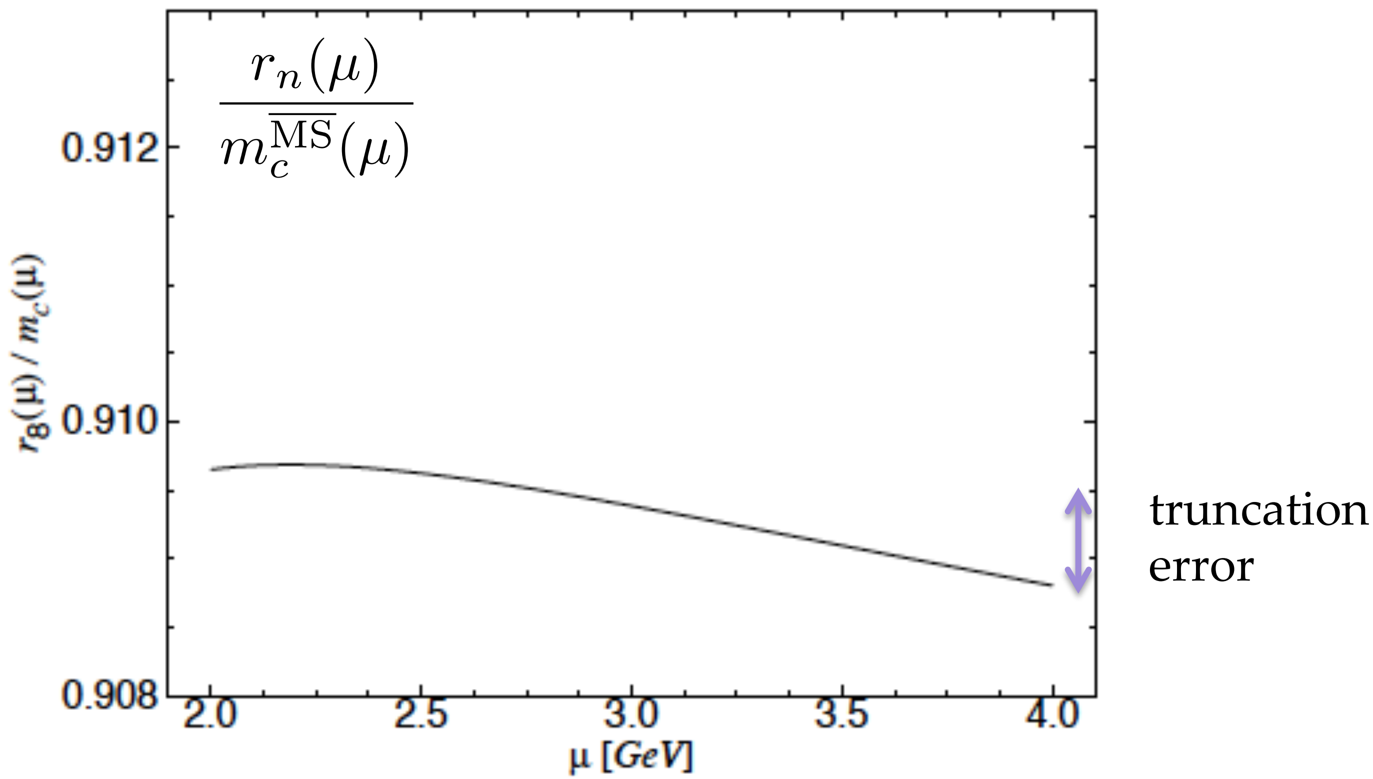
$$R_n = \frac{am_{\eta_c}^{(\mathrm{exp})}}{2a\bar{m}_c(\mu)} r_n(\mu; m_c(\mu), \alpha_s(\mu))$$

- Determine two parameters with the equation of several n.
- Use the pseudo-scalar channel. Exp data do not exist, but the correspondence between lattice and perturbation theory is valid.

LHS: continuum limit

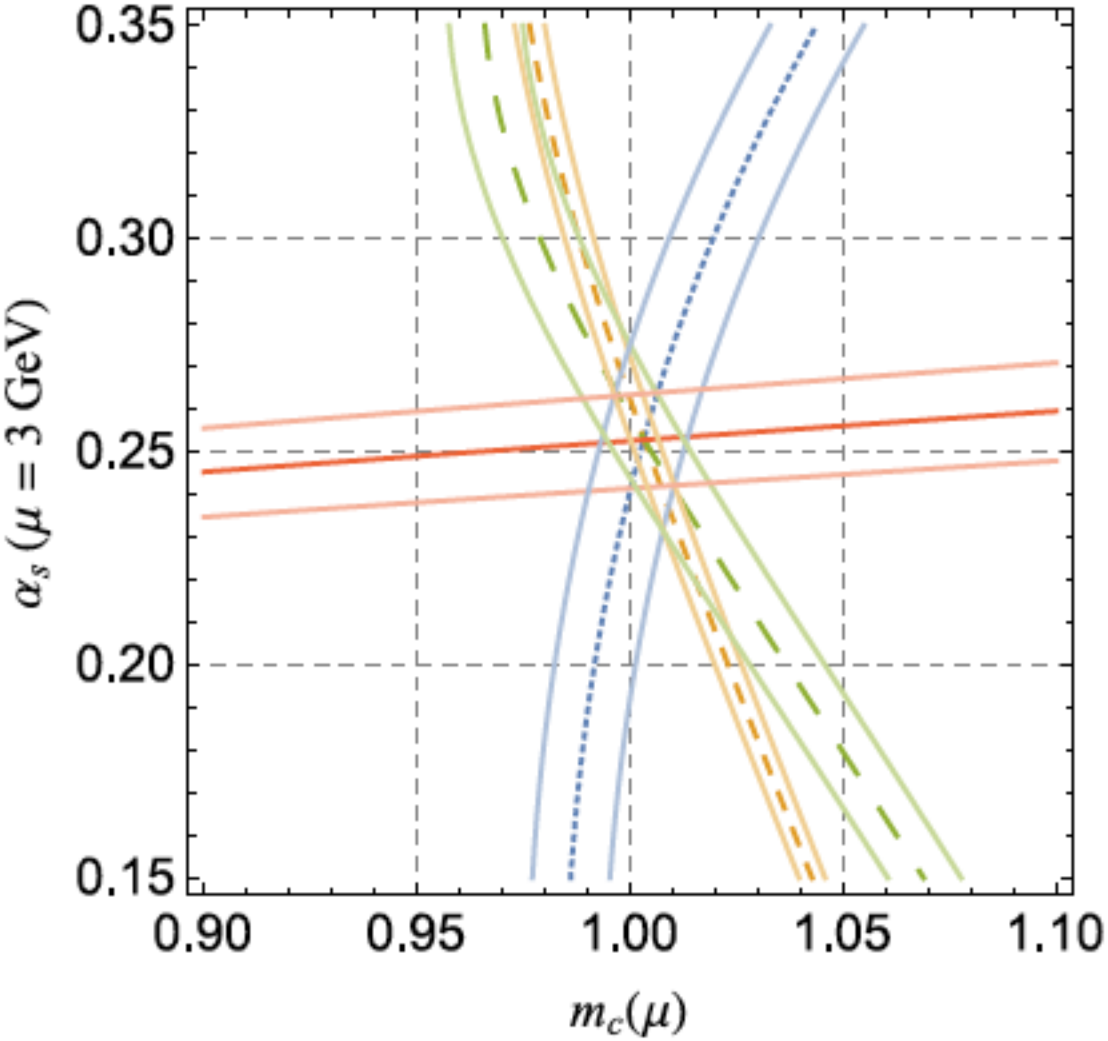


RHS: truncation error of purturbative expansion



Included up to $O(\alpha_s^3)$.
Also included the variation with $\mu_m \neq \mu_\alpha$.

$$R_n = \frac{m_{\eta_c}^{\mathrm{exp}}}{2m_c^{\overline{\mathrm{MS}}}} r_n(m_c^{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$



Determine
$m_c(\mu)$, $\alpha_s(\mu)$, $\langle G_{\mu\nu}^2\rangle$
with 3 moments.

$\alpha_s(\mu)$ is consistent, but its error is not competitive.

m_c(3 GeV) = 1.003(10) GeV

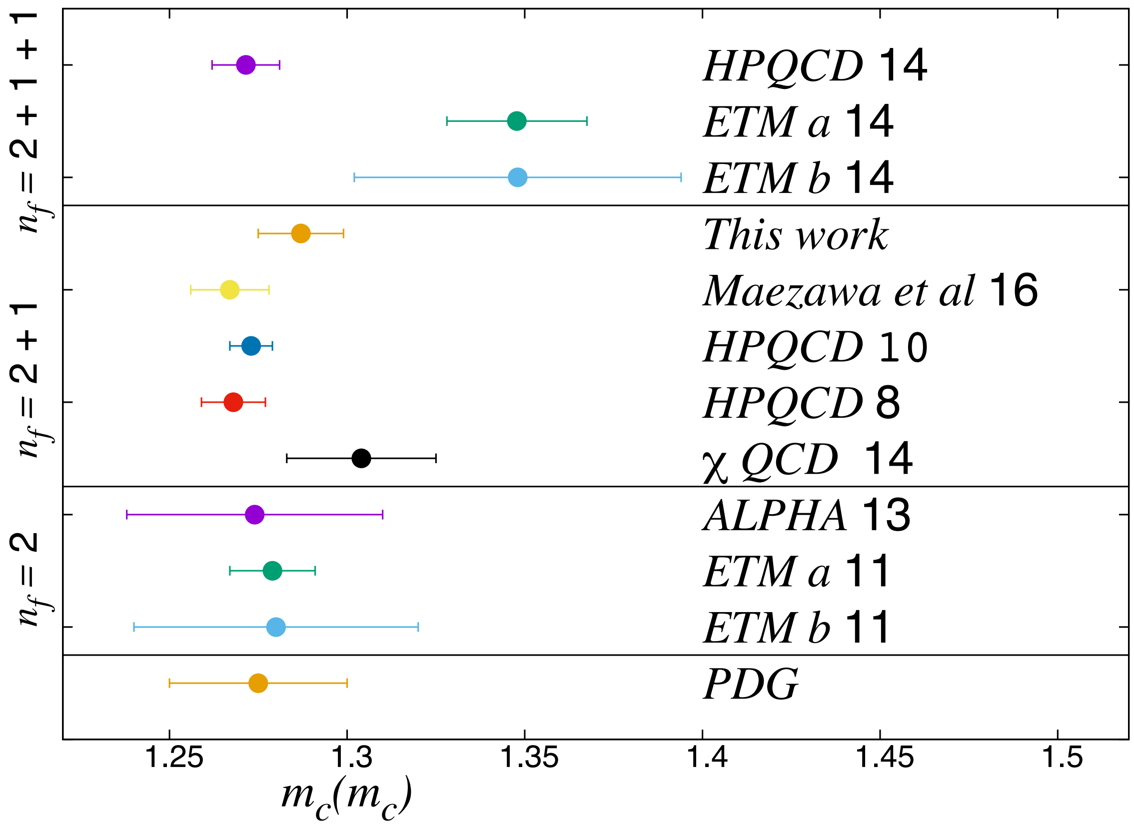
Errors

		pert	$t_0^{1/2}$	stat	$O(a^4)$	vol	$m_{\eta_c}^{\rm exp}$	disc	EM
m_c(3GeV) [GeV]	1.0033(96)	(77)	(49)	(4)	(30)	(4)	(3)	(4)	(6)
α_s(3GeV)	0.2528(127)	(120)	(32)	(2)	(26)	(1)	(0)	(0)	(1)

Dominant = truncation of perturbative expansion

lattice scale
Δa ~ 1%

discretization effect



$n_f = 2+1+1$
$n_f = 2+1$
$n_f = 2$
HPQCD 14
ETM a 14
ETM b 14
This work
Maezawa et al 16
HPQCD 10
HPQCD 8
χ QCD 14
ALPHA 13
ETM a 11
ETM b 11
PDG
1.25
1.3
1.35
1.4
1.45
1.5
$m_c(m_c)$

3. Application to particle phenomenology

3.2 pion and kaon physics

Chiral symmetry breaking

- u, d, s quark masses < 300 MeV
  - Spontaneous symmetry breaking $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$
  - π, K mesons = Nambu-Goldstone bosons

- Effective theory = chiral perturbation theory

$$L_2 = \frac{f^2}{4} \mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{\Sigma}{2} \mathrm{Tr}\left(m U^\dagger + U m^\dagger\right),$$

$$U = \exp\left(\frac{i\tau^a \pi^a}{f}\right)$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

  - Interactions of pions are restricted by symmetry.

Chiral perturbation theory

- Expansion in terms of pion momentum and mass

$$L_2 = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a - \frac{m_\pi^2}{2}\pi^a \pi^a + \frac{m_\pi^2}{24 f^2}(\pi^a \pi^a)^2$$
$$+ \frac{1}{6 f^2}\left[(\pi^a \partial_\mu \pi^a)(\pi^b \partial^\mu \pi^b) - (\pi^a \pi^a)(\partial_\mu \pi^b \partial^\mu \pi^b)\right] + \ldots$$

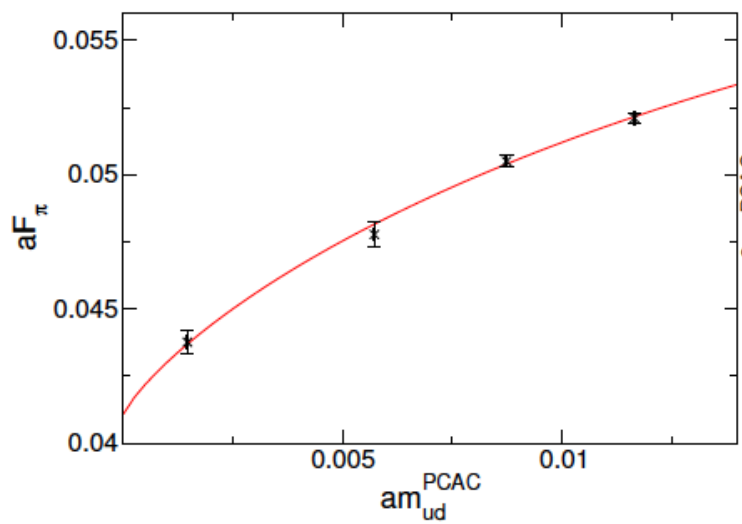
  - Derivative couplings: more reliable for small momenta
  - Loop integral induces higher dimensional Ops (non-renormalizable)
  - Systematic expansion is possible. More parameters (Low Energy Constants) for higher orders: #LO = 2, #NLO = 10. Need to be determined elsewhere.

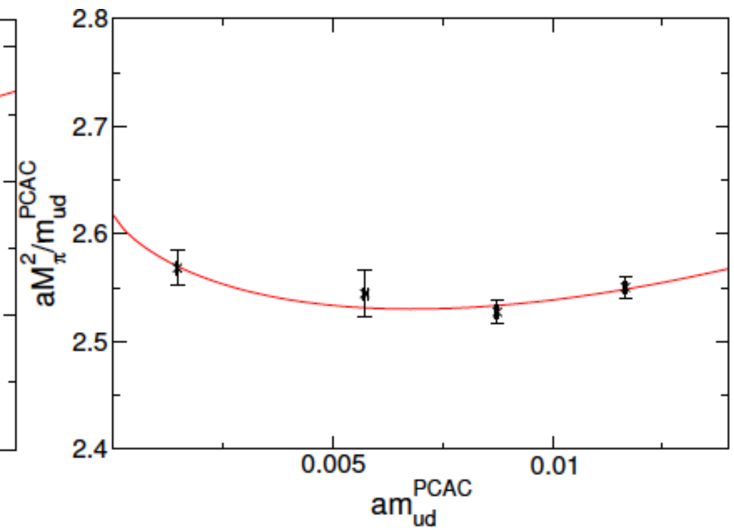
Validate LQCD, and determine LEC.

Consistency with χPT

- Quark mass dependence

BMW (2011)





$$M_\pi^2 = M^2\left\{1+\frac{1}{2}x\ln\frac{M^2}{\Lambda_3^2}+\frac{17}{8}x^2\left(\ln\frac{M^2}{\Lambda_M^2}\right)^2+x^2k_M+O(x^3)\right\}$$

$$F_\pi = F\left\{1-x\ln\frac{M^2}{\Lambda_4^2}-\frac{5}{4}x^2\left(\ln\frac{M^2}{\Lambda_F^2}\right)^2+x^2k_F+O(x^3)\right\},$$

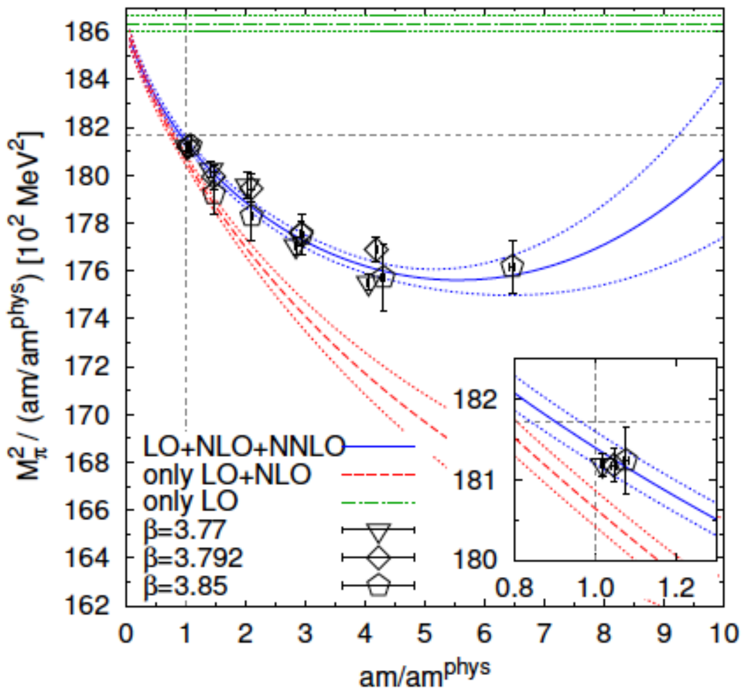
$$x \equiv M^2/(4\pi F)^2$$

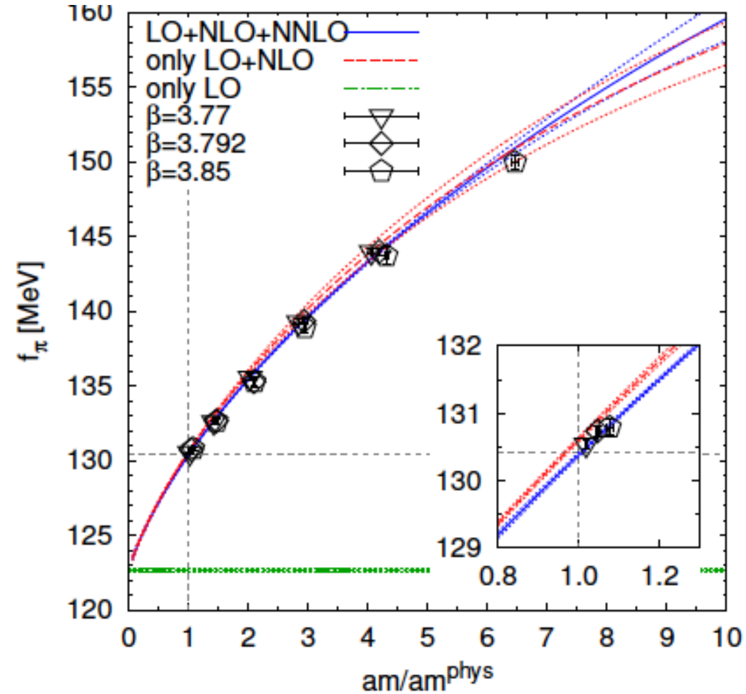
$$M^2 \equiv B(m_1+m_2)$$

Consistency with χPT

- Quark mass dependence

BMW (2014)





NLO and NNLO need to be included to describe the lattice data.

Light quark mass

- Quark mass can be extracted.
  - So far, the bare mass on the lattice.
  - Pole mass doesn't make sense (perturbation theory doesn't converge).

  - Common definition is the MSbar (at 2 GeV); Renormalization factor needs to be calculated.

$$\bar{m}(2\ \mathrm{GeV}) = Z_m(2\ \mathrm{GeV}, 1/a) m^{\mathrm{lat}}$$

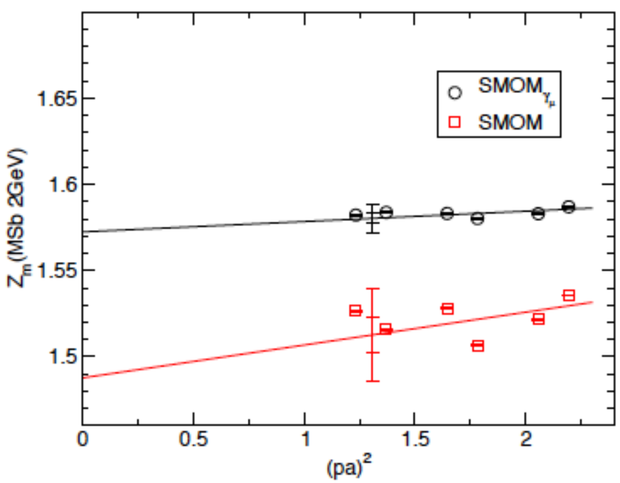
Using perturbation theory, or partly non-perturbatively.

Renormalization

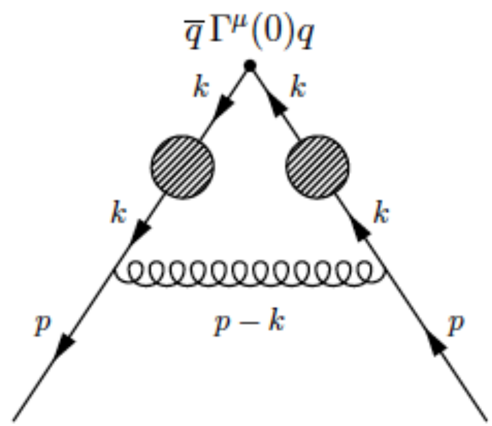
- Use some intermediate scheme to match to MSbar.
  - Ex. RI/MOM scheme, for quark vertex

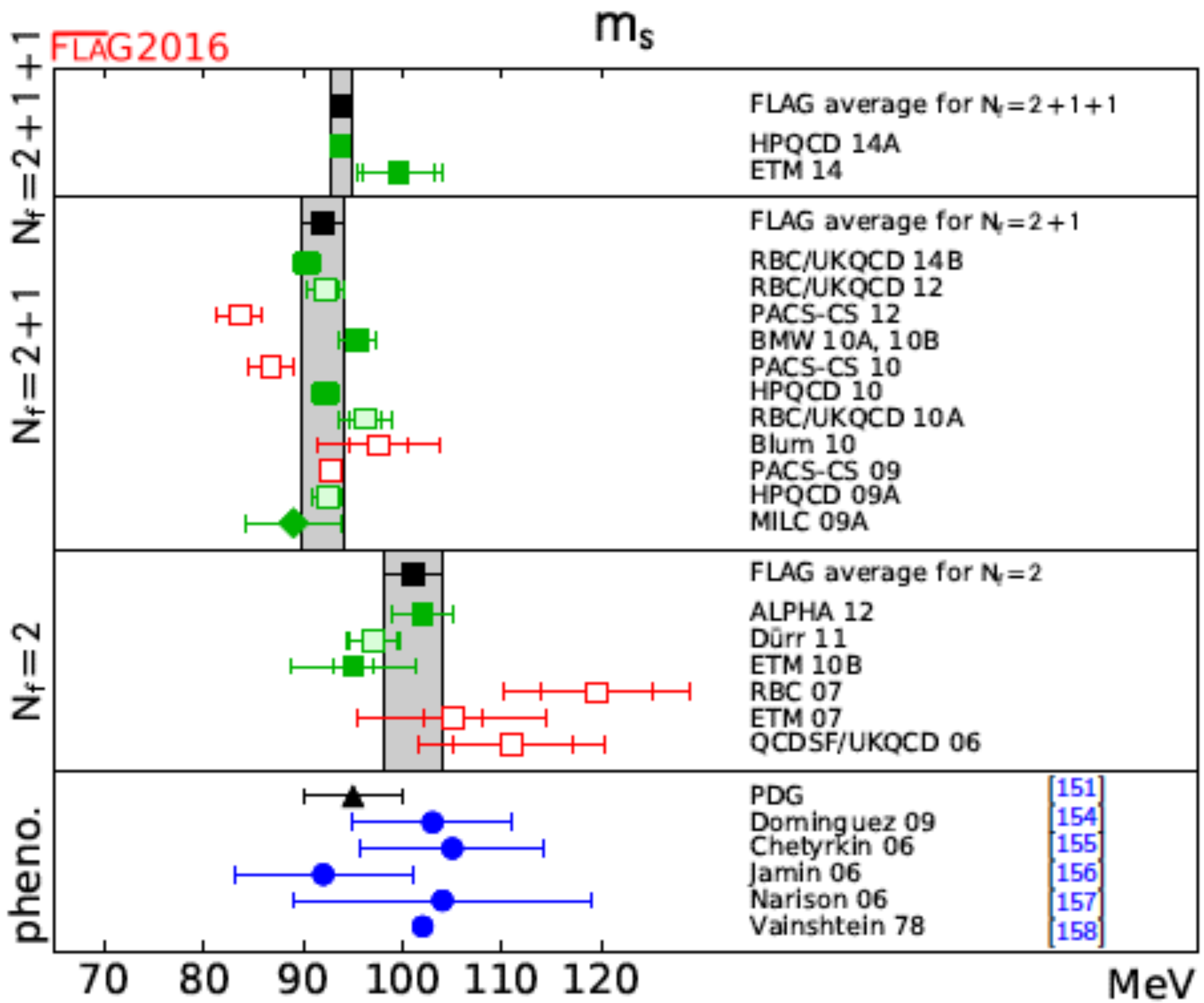
$$\frac{Z_V}{Z_q}\frac{1}{48}\mathrm{Tr}[\Pi_{V_\mu}\cdot\gamma_\mu]=1. \qquad \frac{1}{Z_q}\frac{1}{12}\mathrm{Tr}\left[-i\frac{\partial}{\partial \not p}S^{-1}(p)\right]\Bigg|_{p^2=\mu^2}=1.$$

  - Can be calculated by both MSbar and lattice.

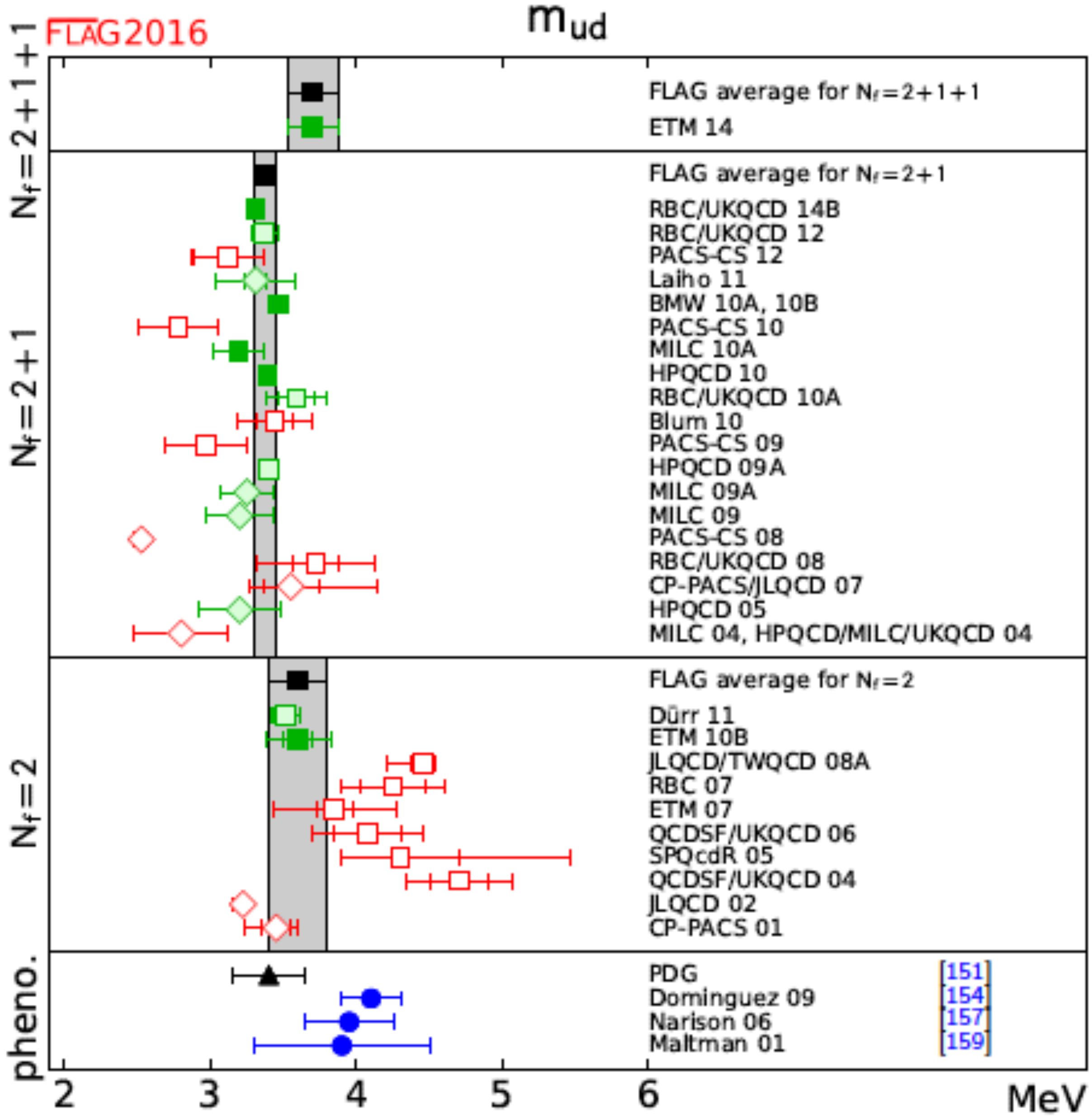


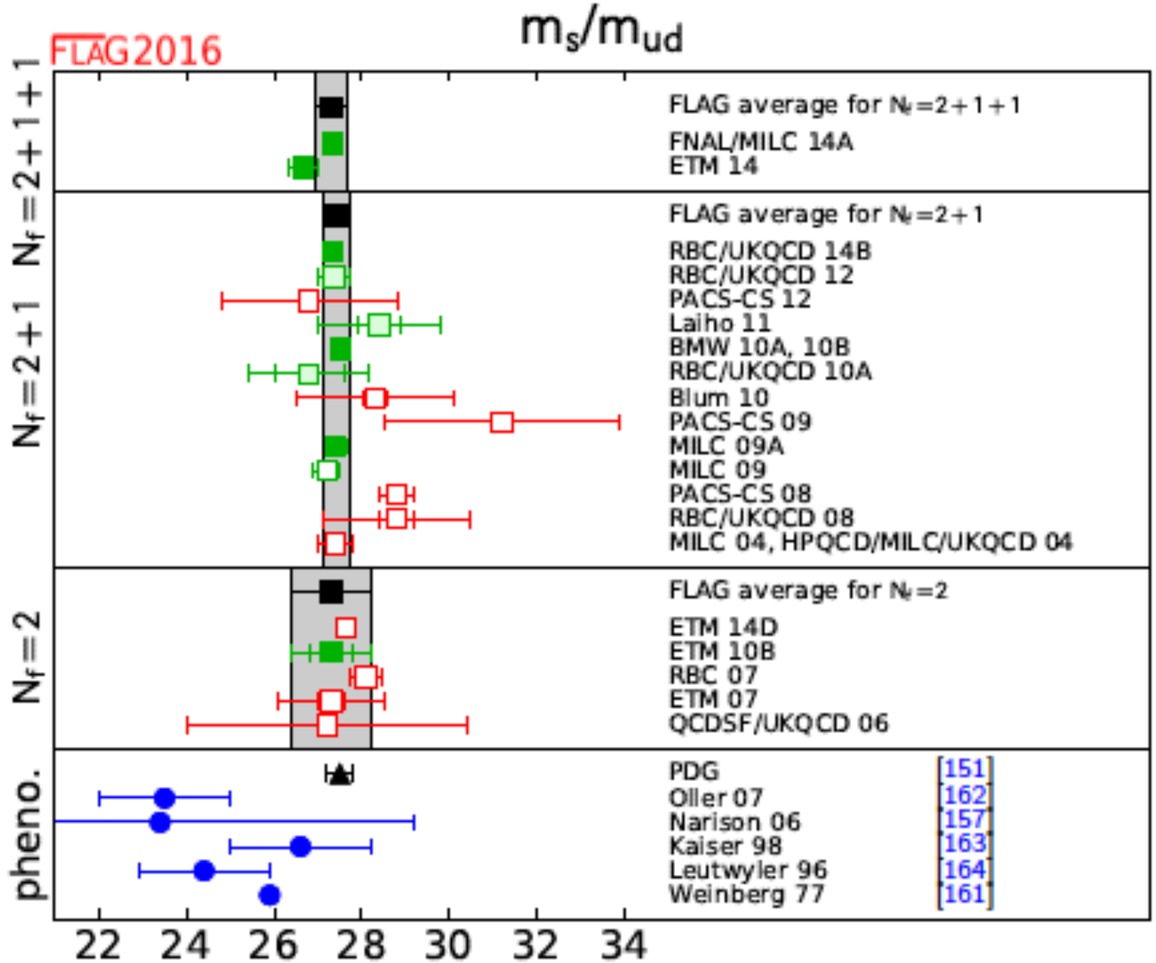
RBC (2010)





FLAG2016
m_s
$N_f=2+1+1$
FLAG average for $N_f=2+1+1$
HPQCD 14A
ETM 14
$N_f=2+1$
FLAG average for $N_f=2+1$
RBC/UKQCD 14B
RBC/UKQCD 12
PACS-CS 12
BMW 10A, 10B
PACS-CS 10
HPQCD 10
RBC/UKQCD 10A
Blum 10
PACS-CS 09
HPQCD 09A
MILC 09A
$N_f=2$
FLAG average for $N_f=2$
ALPHA 12
Dürr 11
ETM 10B
RBC 07
ETM 07
QCDSF/UKQCD 06
pheno.
PDG [151]
Dominguez 09 [154]
Chetyrkin 06 [155]
Jamin 06 [156]
Narison 06 [157]
Vainshtein 78 [158]
70
80
90
100
110
120
MeV


FLAG2016
m_{ud}
$N_f=2+1+1$
FLAG average for $N_f=2+1+1$
ETM 14
$N_f=2+1$
FLAG average for $N_f=2+1$
RBC/UKQCD 14B
RBC/UKQCD 12
PACS-CS 12
Laiho 11
BMW 10A, 10B
PACS-CS 10
MILC 10A
HPQCD 10
RBC/UKQCD 10A
Blum 10
PACS-CS 09
HPQCD 09A
MILC 09A
MILC 09
PACS-CS 08
RBC/UKQCD 08
CP-PACS/JLQCD 07
HPQCD 05
MILC 04, HPQCD/MILC/UKQCD 04
$N_f=2$
FLAG average for $N_f=2$
Dürr 11
ETM 10B
JLQCD/TWQCD 08A
RBC 07
ETM 07
QCDSF/UKQCD 06
SPQcdR 05
QCDSF/UKQCD 04
JLQCD 02
CP-PACS 01
pheno.
PDG [151]
Dominguez 09 [154]
Narison 06 [157]
Maltman 01 [159]
2
3
4
5
6
MeV


FLAG2016
m_s/m_{ud}
$N_f=2+1+1$
$N_f=2+1$
$N_f=2$
pheno.
FLAG average for $N_f=2+1+1$
FNAL/MILC 14A
ETM 14
FLAG average for $N_f=2+1$
RBC/UKQCD 14B
RBC/UKQCD 12
PACS-CS 12
Laiho 11
BMW 10A, 10B
RBC/UKQCD 10A
Blum 10
PACS-CS 09
MILC 09A
MILC 09
PACS-CS 08
RBC/UKQCD 08
MILC 04, HPQCD/MILC/UKQCD 04
FLAG average for $N_f=2$
ETM 14D
ETM 10B
RBC 07
ETM 07
QCDSF/UKQCD 06
PDG [151]
Oller 07 [162]
Narison 06 [157]
Kaiser 98 [163]
Leutwyler 96 [164]
Weinberg 77 [161]
22 24 26 28 30 32 34

Pion charge radius

- EM form factor

$$\langle P(p')|J_\mu|P(p)\rangle = (p+p')_\mu F_V^P(t), \quad t=(p-p')^2,$$
$$J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s,$$

  - Charge radius

$$\langle r^2\rangle_V^P = 6\left.\frac{\partial F_V^P(t)}{\partial t}\right|_{t=0},$$

  - Vector meson dominance

$$F_V(t) = \frac{1}{1+t/m_V^2}$$

Pion charge radius

- Three-point function

$$C_{KV_\mu D}(t_x,t_y;\vec{p}) = \sum_{\vec{x},\vec{y}} \langle O_K(t_x,\vec{x}) V_\mu(0) O_D^\dagger(t_y,\vec{y})\rangle e^{-i\vec{p}\cdot\vec{x}}$$

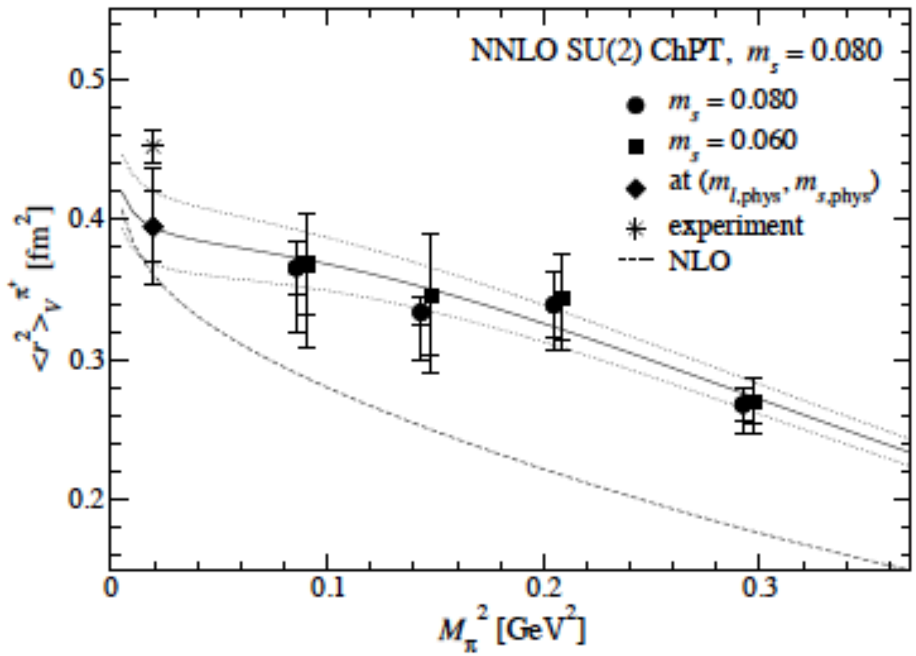
  - inserting complete set of states,

$$C_{KV_\mu D}(t_x,t_y;\vec{p}) = \sum_{i,j} \frac{1}{2m_{D_i} 2E_{K_j}(\vec{p})} e^{-m_{D_i}t_x - E_{K_j}(\vec{p})|t_y|} \times$$

$$\times \langle 0|O_K(t_x,\vec{x})|K_i(\vec{p})\rangle\langle K_i(\vec{p})|V_\mu(0)|D_j(\vec{0})\rangle\langle D_j(\vec{0})|O_D^\dagger(0)|0\rangle$$

Pion charge radius

- Charge radius



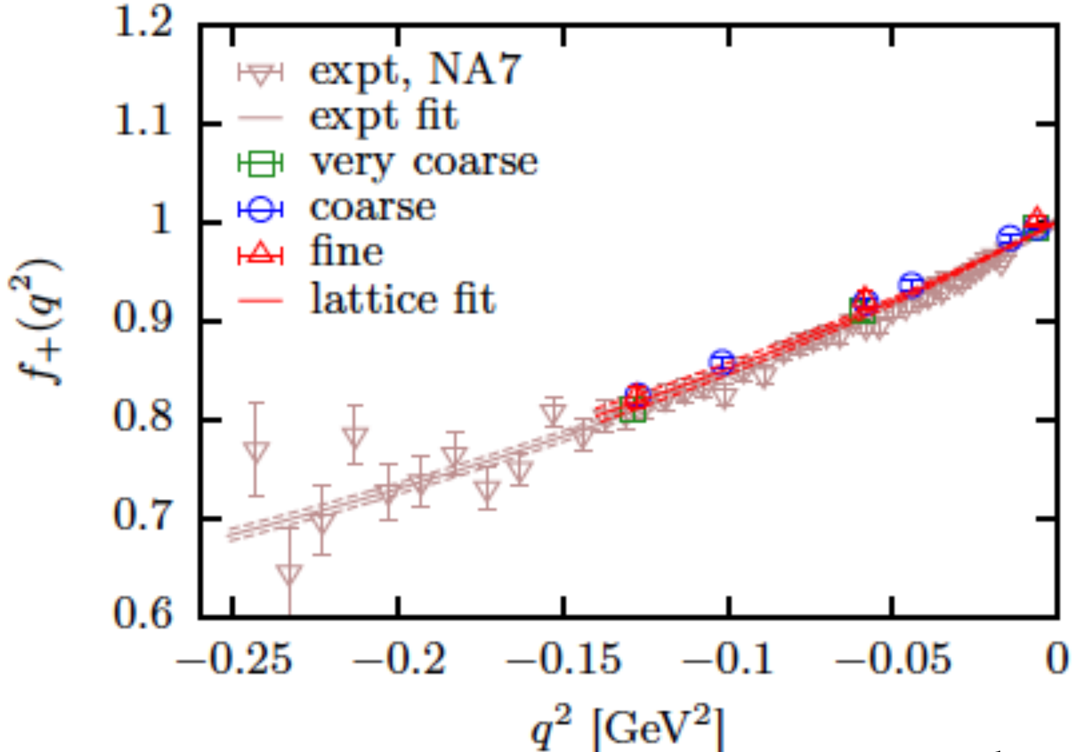
$$\langle r^2\rangle_V^P = 6 \left.\frac{\partial F_V^P(t)}{\partial t}\right|_{t=0},$$

JLQCD (2015)

$$\langle r^2\rangle_V^\pi = \frac{1}{(4\pi F_\pi)^2}\left\{\ln\frac{\Lambda_6^2}{M_\pi^2} - 1 + 2\xi\left(\ln\frac{\Omega_{r_V}^2}{M_\pi^2}\right)^2 + 6\xi\, k_{r_V} + \mathcal{O}(\xi^2)\right\}$$

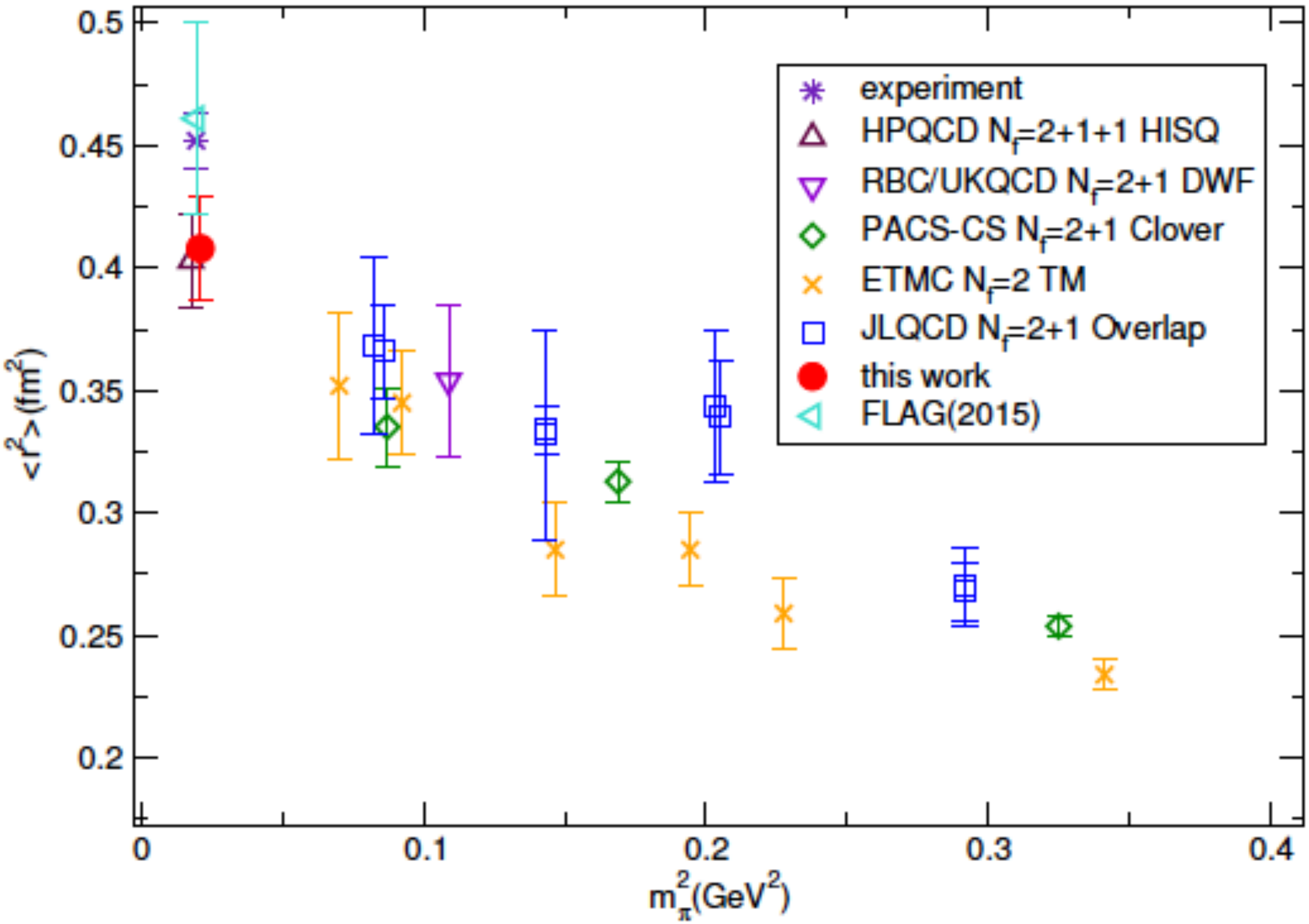
Pion charge radius

HPQCD (2015)



at the physical point

PACS (2016)



experiment
HPQCD N$_f$=2+1+1 HISQ
RBC/UKQCD N$_f$=2+1 DWF
PACS-CS N$_f$=2+1 Clover
ETMC N$_f$=2 TM
JLQCD N$_f$=2+1 Overlap
this work
FLAG(2015)
<r^2> (fm^2)
m$_\pi^2$(GeV^2)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0
0.1
0.2
0.3
0.4

$\pi\pi$ scattering

- Scattering length a_0

$$a_0 \sim \tan \delta_0(k)/k$$

$$\frac{1}{\tan \delta_0(k)} = \frac{4\pi}{k} \cdot \frac{1}{L^3} \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{p_n^2 - k^2} \qquad (\mathbf{p}_n = \mathbf{n} \cdot (2\pi)/L)$$

: SC. phase shift in infinite volume

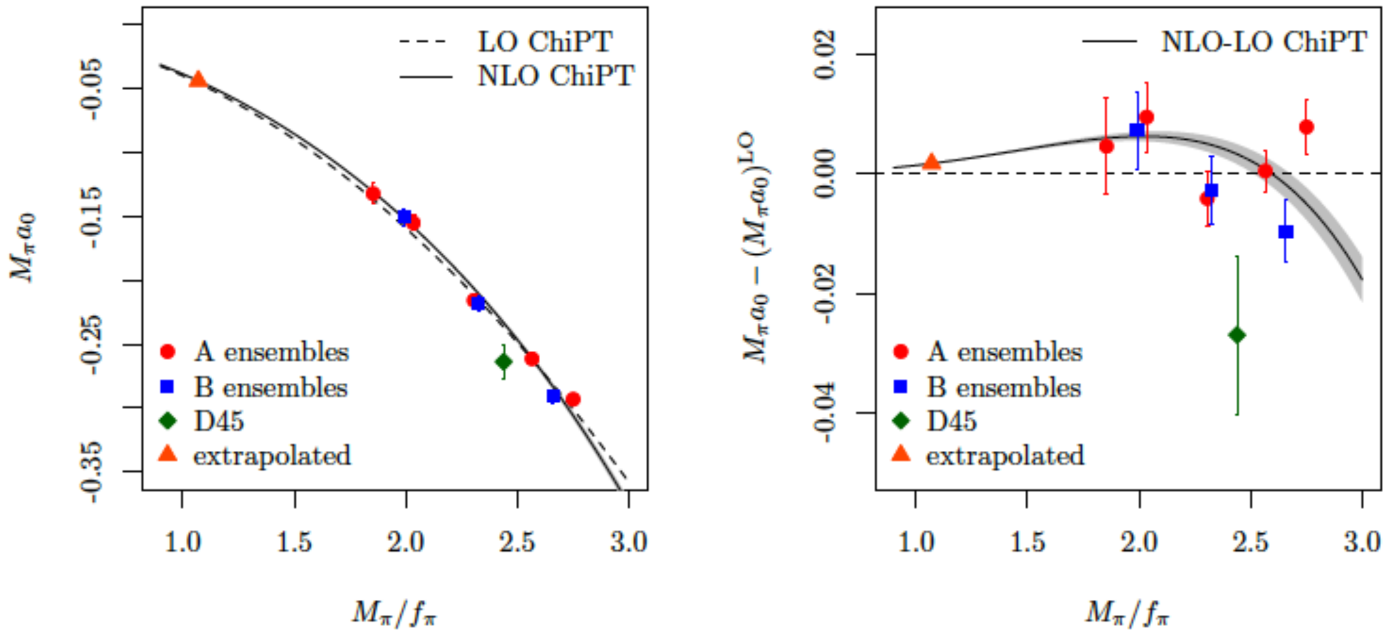
: Lüscher's formula

  - Allowed energy in a finite box is limited. Contains the info of scattering phase shift.

$\pi\pi$ scattering

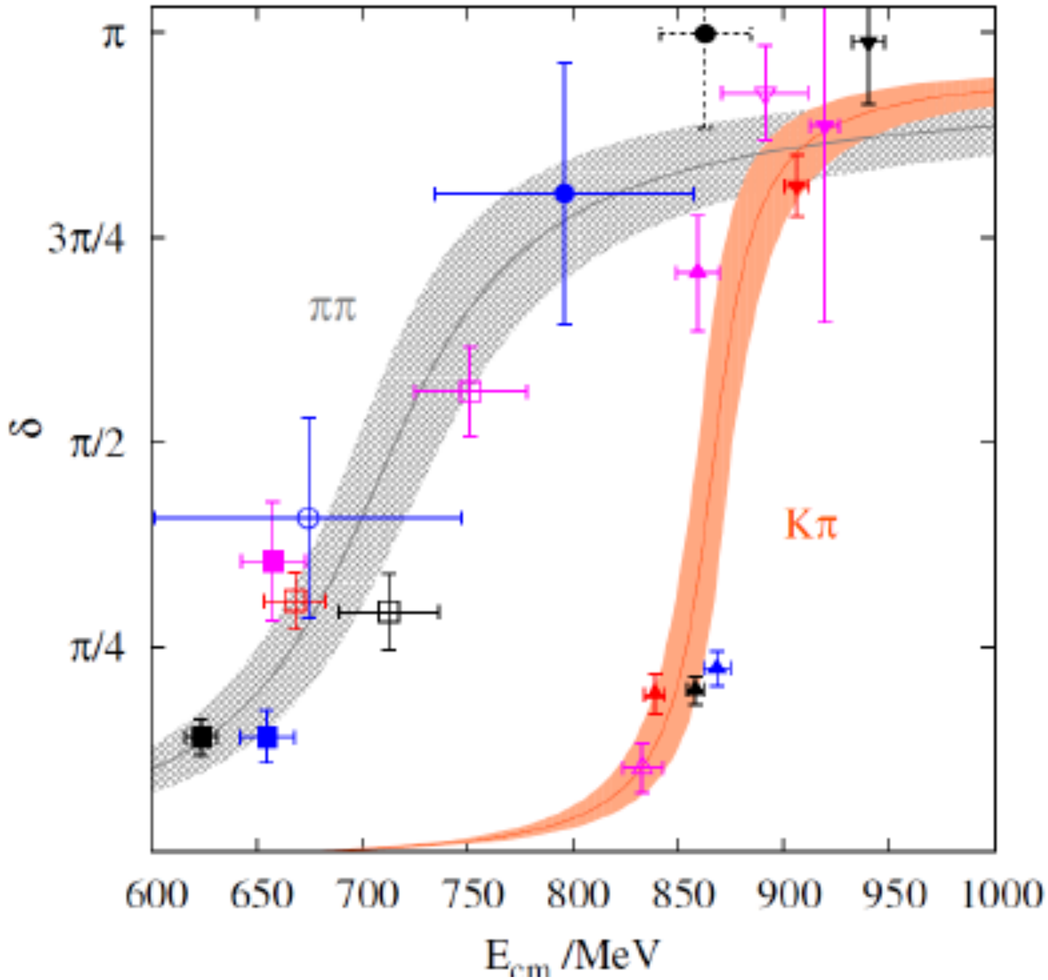
- Scattering length a_0

I=2 channel. ETMC (2015)



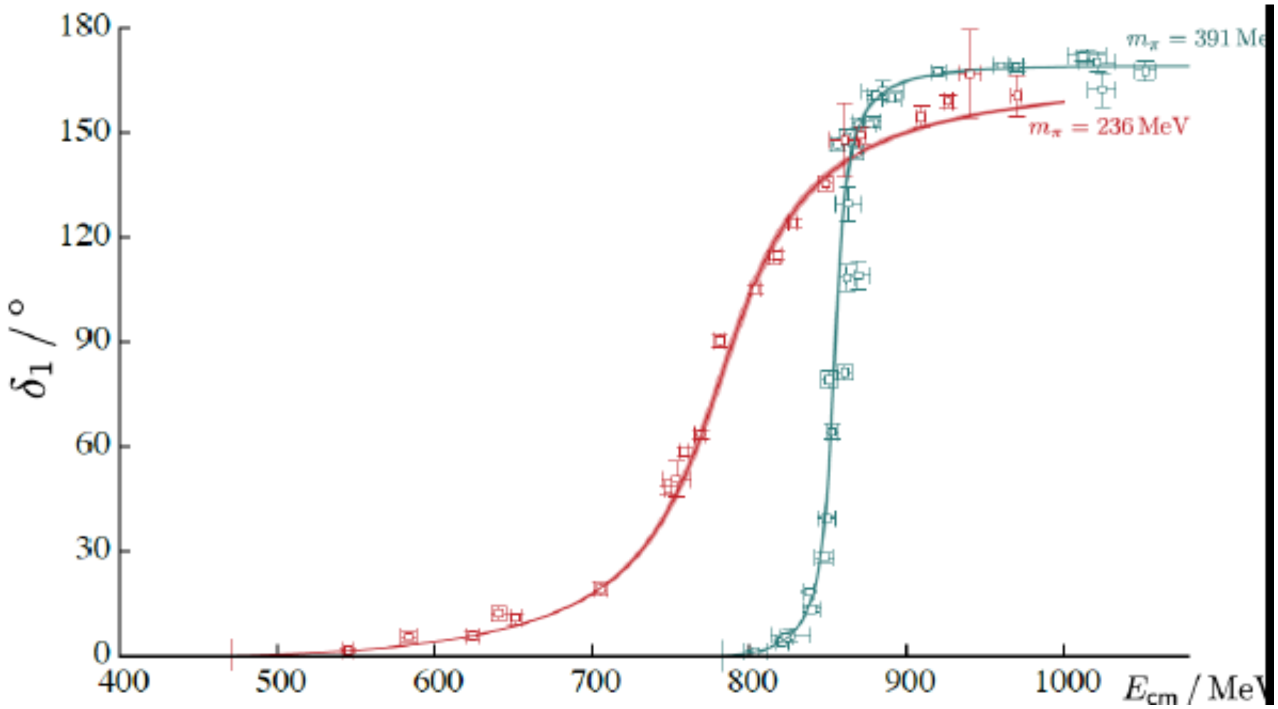
$$M_\pi a_0 = -\frac{M_\pi^2}{8\pi f_\pi^2}\left\{1+\frac{M_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\frac{M_\pi^2}{f_\pi^2}-1-\ell_{\pi\pi}(\mu_R=f_{\pi,\mathrm{phys}})\right]\right\}$$

$\pi\pi$ scattering



I=1 channel.
Bali et al. (2016)

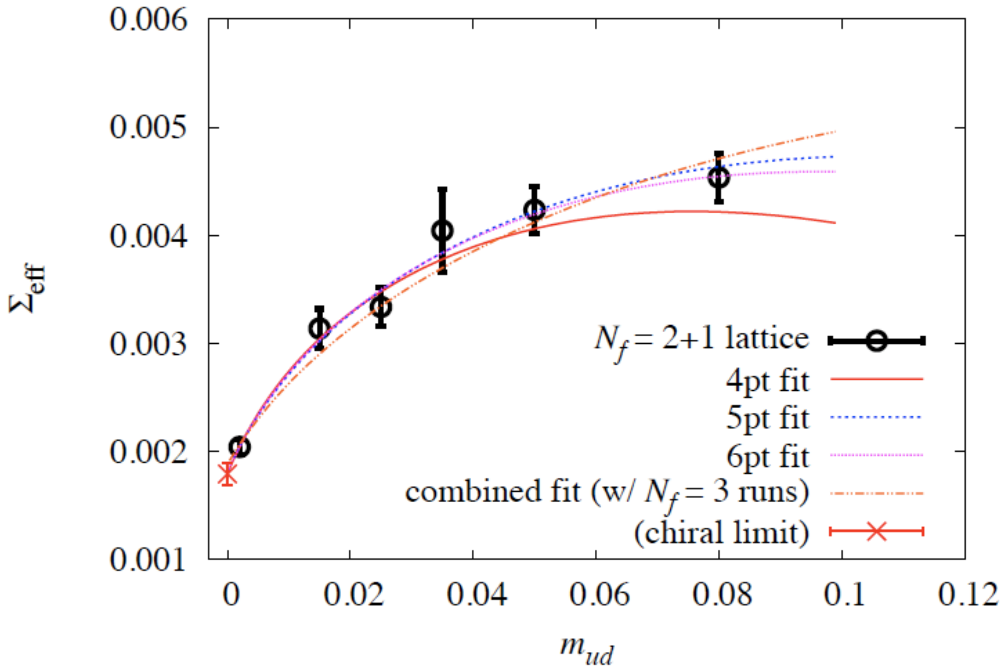
$\pi\pi$ scattering



I=1 channel.
HSC (2016)

Chiral condensate

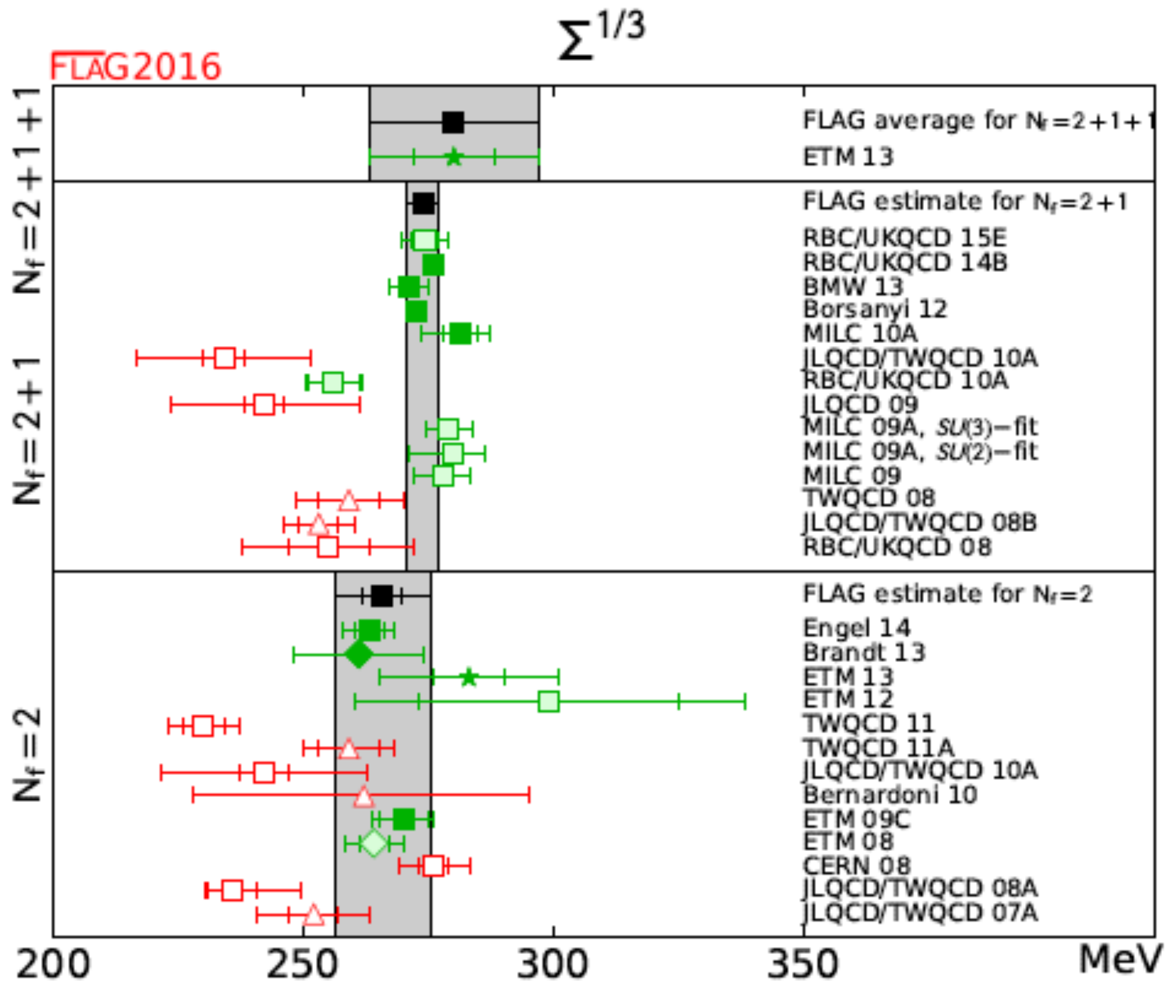
- Strength of symmetry breaking



JLQCD (2012)

$$\Sigma(m_{ud}, m_s) = \Sigma(0, m_s)\left[1 - \frac{3M_\pi^2}{32\pi^2 F^2}\ln\frac{M_\pi^2}{\mu^2} + \frac{32 L_6 M_\pi^2}{F^2}\right]$$

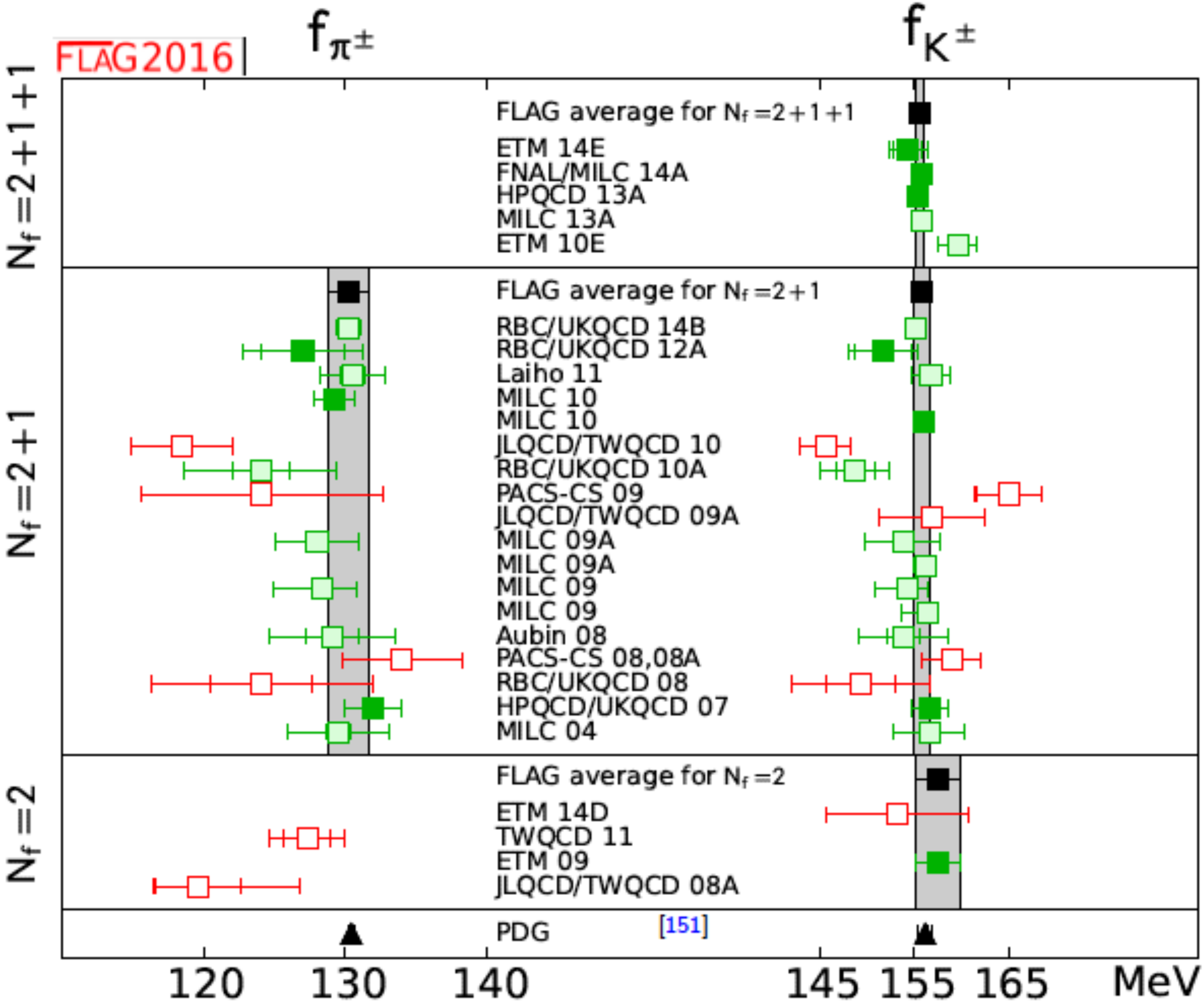
Chiral condensate



FLAG averages

Flavor Lattice Averaging Group 3, arXiv:1607.00299

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_s[MeV]	3.1.3	93.9(1.1)	[4, 5]	92.0(2.1)	[6–10]	101(3)	[11, 12]
m_{ud}[MeV]	3.1.3	3.70(17)	[4]	3.373(80)	[7–10, 13]	3.6(2)	[11]
m_s/m_{ud}	3.1.4	27.30(34)	[4, 14]	27.43(31)	[7, 8, 10, 15]	27.3(9)	[11]
m_u[MeV]	3.1.5	2.36(24)	[4]	2.16(9)(7)	‡	2.40(23)	[16]
m_d[MeV]	3.1.5	5.03(26)	[4]	4.68(14)(7)	‡	4.80(23)	[16]
m_u/m_d	3.1.5	0.470(56)	[4]	0.46(2)(2)	‡	0.50(4)	[16]
$\overline{m}_c$(3 GeV)[GeV]	3.2.3	0.996(25)	[4, 5]	0.987(6)	[9, 17]	1.03(4)	[11]
m_c/m_s	3.2.4	11.70(6)	[4, 5, 14]	11.82(16)	[17, 18]	11.74(35)	[11]
$\overline{m}_b(\overline{m}_b)$[GeV]	3.3	4.190(21)	[5, 19]	4.164(23)	[9]	4.256(81)	[20, 21]
$f_+(0)$	4.3	0.9704(24)(22)	[22]	0.9677(27)	[23, 24]	0.9560(57)(62)	[25]
$f_{K^\pm}/f_{\pi^\pm}$	4.3	1.193(3)	[14, 26, 27]	1.192(5)	[28–31]	1.205(6)(17)	[32]
$f_{\pi^\pm}$[MeV]	4.6			130.2(1.4)	[28, 29, 31]		
$f_{K^\pm}$[MeV]	4.6	155.6(4)	[14, 26, 27]	155.9(9)	[28, 29, 31]	157.5(2.4)	[32]
$\Sigma^{1/3}$[MeV]	5.2.1	280(8)(15)	[33]	274(3)	[10, 13, 34, 35]	266(10)	[33, 36–38]
F_π/F	5.2.1	1.076(2)(2)	[39]	1.064(7)	[10, 29, 34, 35, 40]	1.073(15)	[36–38, 41]
$\bar{\ell}_3$	5.2.2	3.70(7)(26)	[39]	2.81(64)	[10, 29, 34, 35, 40]	3.41(82)	[36, 37, 41]
$\bar{\ell}_4$	5.2.2	4.67(3)(10)	[39]	4.10(45)	[10, 29, 34, 35, 40]	4.51(26)	[36, 37, 41]
$\bar{\ell}_6$	5.2.2					15.1(1.2)	[37, 41]
$\hat{B}_K$	6.1	0.717(18)(16)	[42]	0.7625(97)	[10, 43–45]	0.727(22)(12)	[46]



FLAG2016
$f_{\pi^\pm}$
$f_{K^\pm}$
$N_f=2+1+1$
$N_f=2+1$
$N_f=2$
FLAG average for $N_f=2+1+1$
ETM 14E
FNAL/MILC 14A
HPQCD 13A
MILC 13A
ETM 10E
FLAG average for $N_f=2+1$
RBC/UKQCD 14B
RBC/UKQCD 12A
Laiho 11
MILC 10
MILC 10
JLQCD/TWQCD 10
RBC/UKQCD 10A
PACS-CS 09
JLQCD/TWQCD 09A
MILC 09A
MILC 09A
MILC 09
MILC 09
Aubin 08
PACS-CS 08,08A
RBC/UKQCD 08
HPQCD/UKQCD 07
MILC 04
FLAG average for $N_f=2$
ETM 14D
TWQCD 11
ETM 09
JLQCD/TWQCD 08A
PDG [151]
120
130
140
145
155
165
MeV

FLAG averages

Flavor Lattice Averaging Group 3, arXiv:1607.00299

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
f_D[MeV]	7.1	212.15(1.45)	[14, 27]	209.2(3.3)	[47, 48]	208(7)	[20]
f_{D_s}[MeV]	7.1	248.83(1.27)	[14, 27]	249.8(2.3)	[17, 48, 49]	250(7)	[20]
f_{D_s}/f_D	7.1	1.1716(32)	[14, 27]	1.187(12)	[47, 48]	1.20(2)	[20]
$f_+^{D\pi}(0)$	7.2			0.666(29)	[50]		
$f_+^{DK}(0)$	7.2			0.747(19)	[51]		
f_B[MeV]	8.1	186(4)	[52]	192.0(4.3)	[48, 53–56]	188(7)	[20, 57, 58]
f_{B_s}[MeV]	8.1	224(5)	[52]	228.4(3.7)	[48, 53–56]	227(7)	[20, 57, 58]
f_{B_s}/f_B	8.1	1.205(7)	[52]	1.201(16)	[48, 53–56]	1.206(23)	[20, 57, 58]
$f_{B_d}\sqrt{\hat{B}_{B_d}}$[MeV]	8.2			219(14)	[54, 59]	216(10)	[20]
$f_{B_s}\sqrt{\hat{B}_{B_s}}$[MeV]	8.2			270(16)	[54, 59]	262(10)	[20]
$\hat{B}_{B_d}$	8.2			1.26(9)	[54, 59]	1.30(6)	[20]
$\hat{B}_{B_s}$	8.2			1.32(6)	[54, 59]	1.32(5)	[20]
ξ	8.2			1.239(46)	[54, 60]	1.225(31)	[20]
B_{B_s}/B_{B_d}	8.2			1.039(63)	[54, 60]	1.007(21)	[20]

Quantity	Sec.	$N_f = 2+1$ and $N_f = 2+1+1$	Refs.
$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z)$	9.9	0.1182(12)	[5, 9, 61–63]
$\Lambda^{(5)}_{\overline{\mathrm{MS}}}$[MeV]	9.9	211(14)	[5, 9, 61–63]

3. Application to particle phenomenology

3.3 nucleon properties

Nucleon form factor

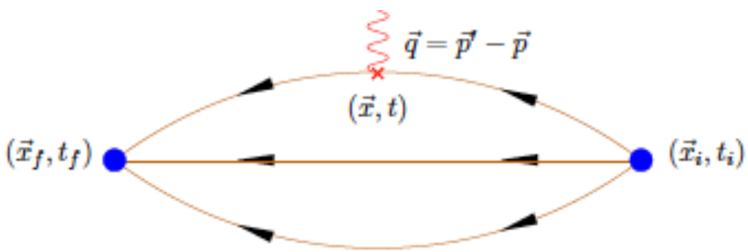
- Matrix elements (vector)

$$\langle N(p',s')|j^\mu|N(p,s)\rangle = \left(\frac{m_N^2}{E_N(\mathbf{p}')E_N(\mathbf{p})}\right)^{1/2} \bar{u}(p',s')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_N}F_2(q^2)\right]u(p,s)$$

  - Electromagnetic form factors

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{(2m_N)^2}F_2(q^2)$$

$$G_M(q^2) = F_1(q^2) + F_2(q^2)$$



Nucleon form factor

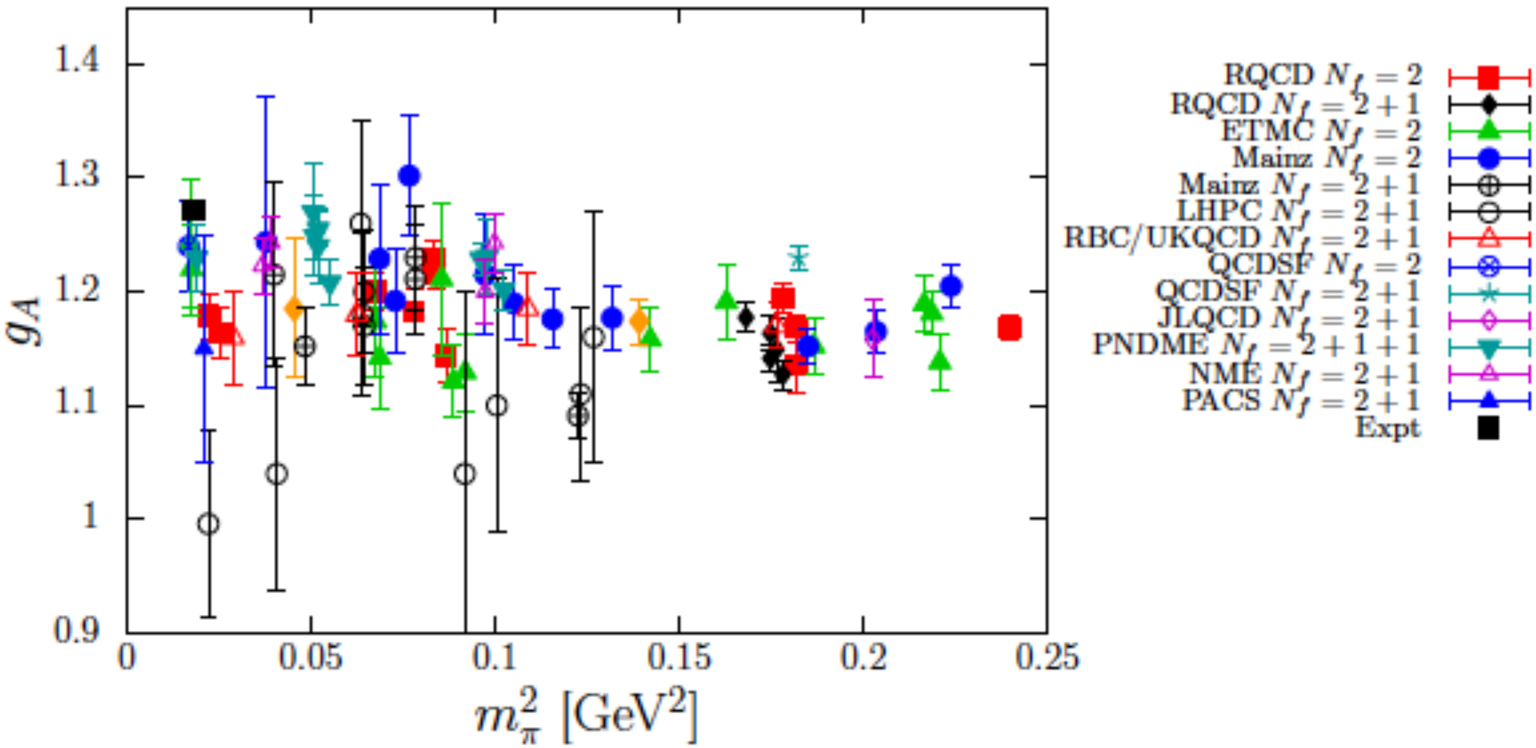
- Matrix elements (axial-vector)

$$\langle N(p',s')|A_\mu^3|N(p,s)\rangle = \frac{i}{2}\left(\frac{m_N^2}{E_N(\mathbf{p}')E_N(\mathbf{p})}\right)^{1/2}\bar{u}(p',s')\left[\gamma_\mu\gamma_5 G_A(q^2)+\frac{q_\mu\gamma_5}{2m_N}G_P(q^2)\right]u(p,s)$$

  - axial charge $g_A = G_A(0)$
    - Well determined experimentally through the beta decay.

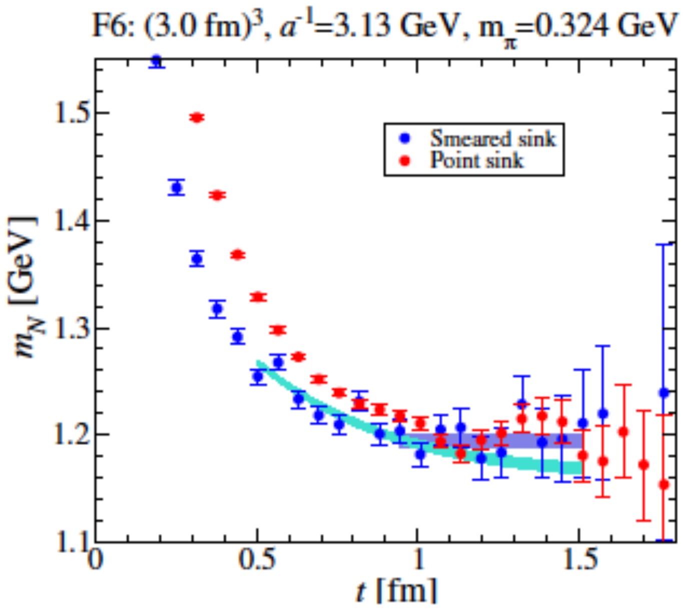
axial charge g_A

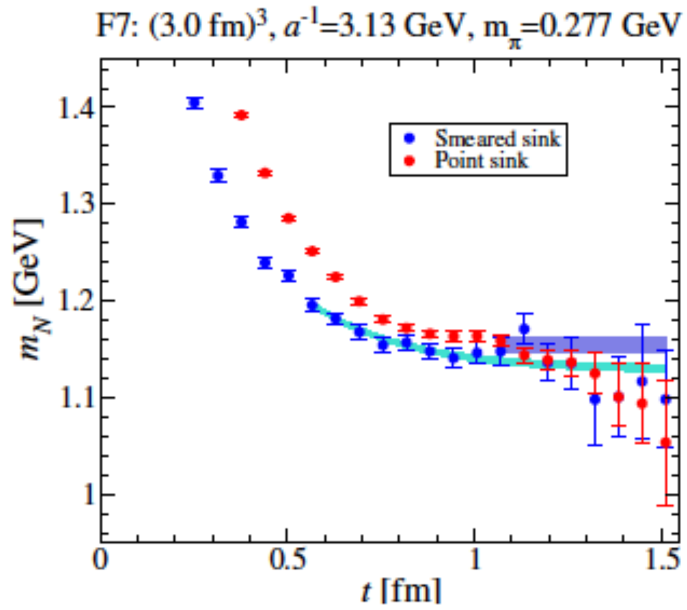
- A benchmark of lattice QCD calculation



Ground state?

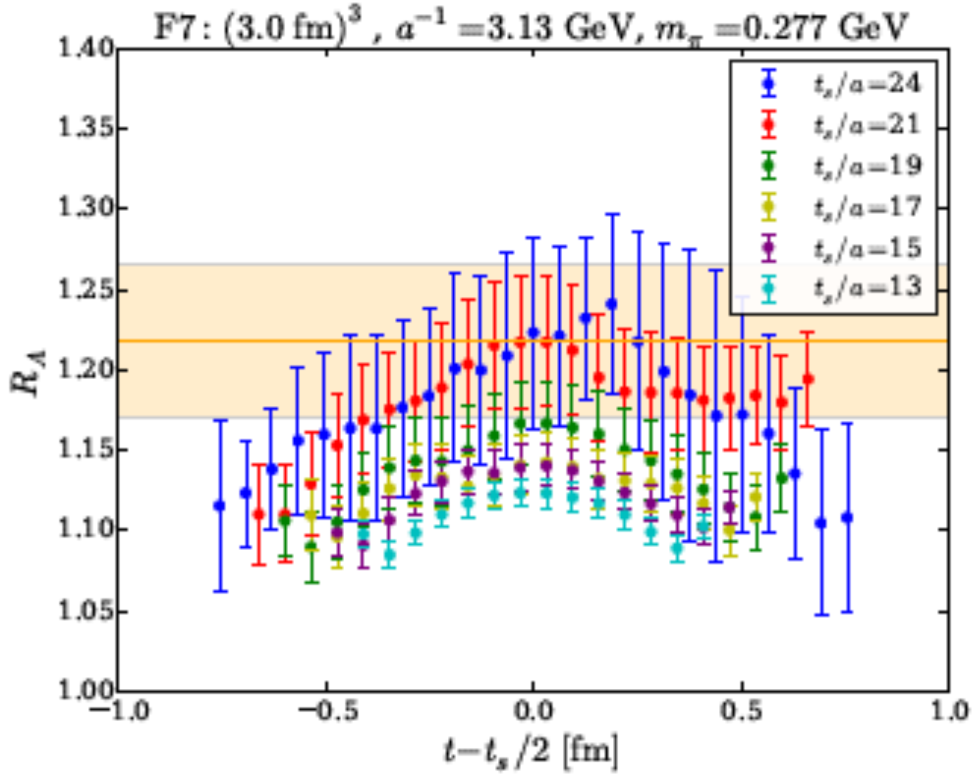
Mainz (2016)





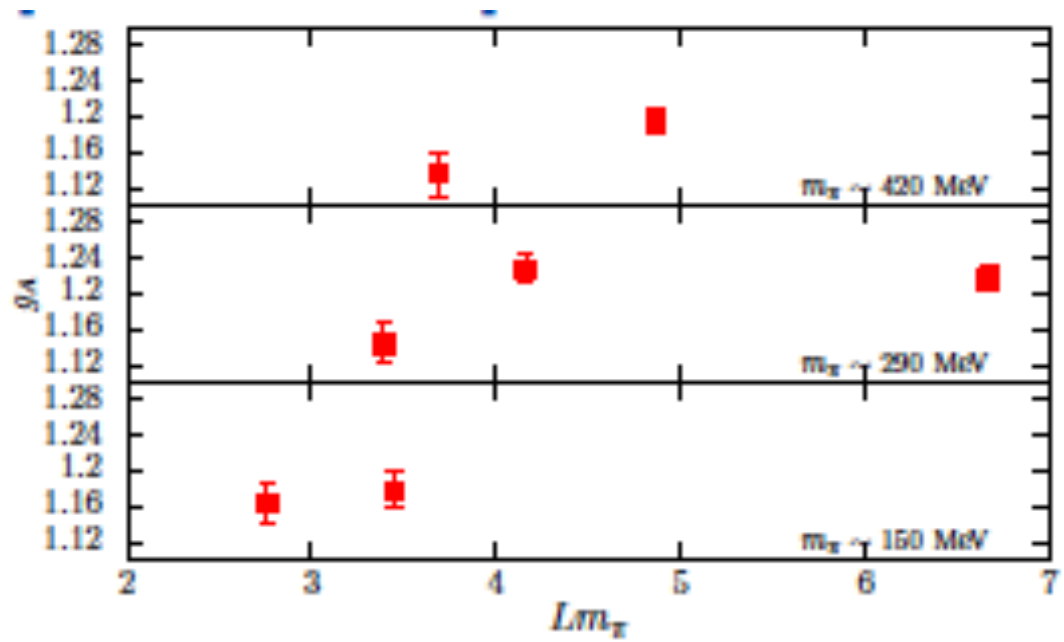
Ground state?

Mainz (2016)

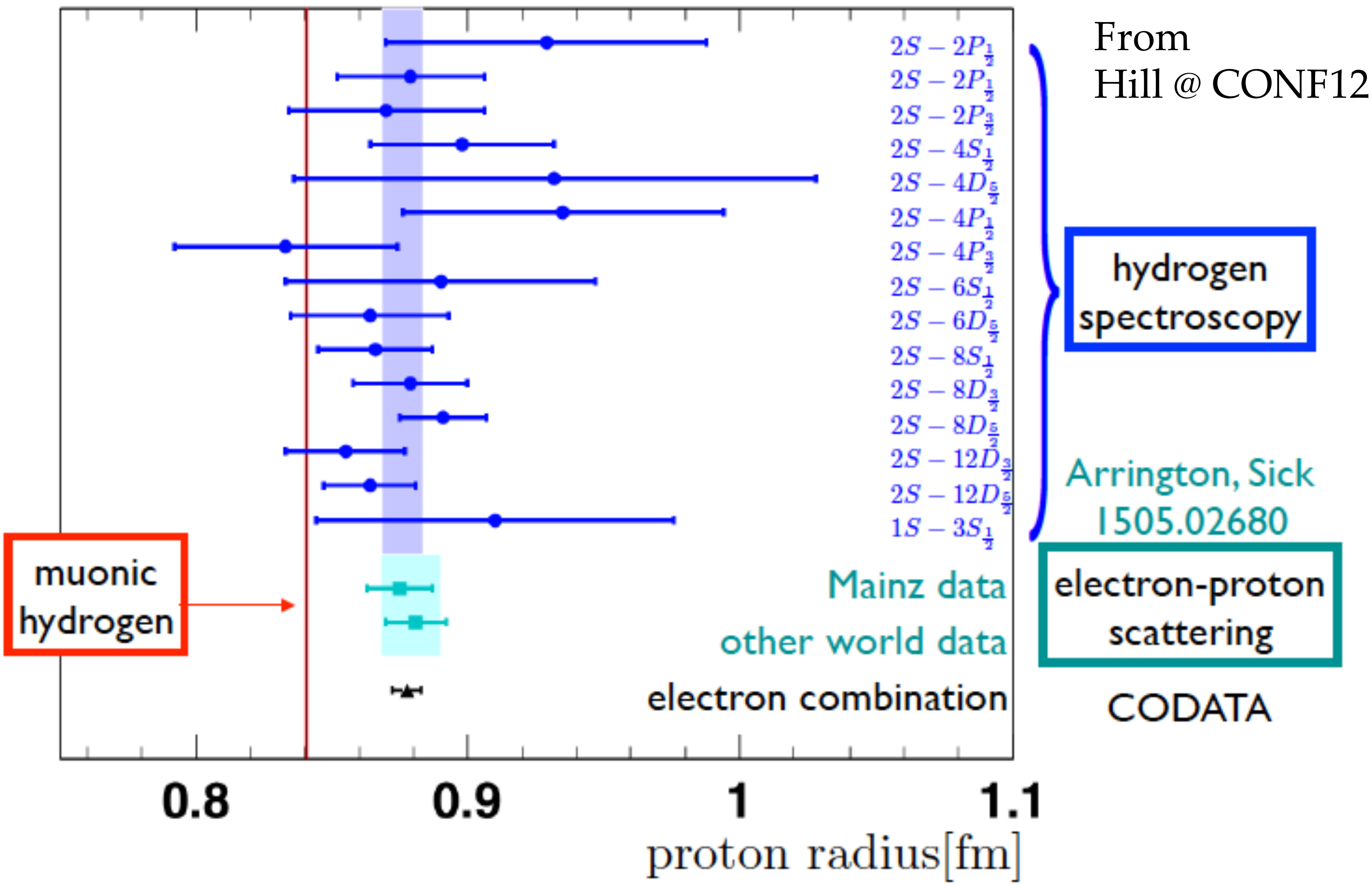


Finite volume?

Bali et al. (2014)



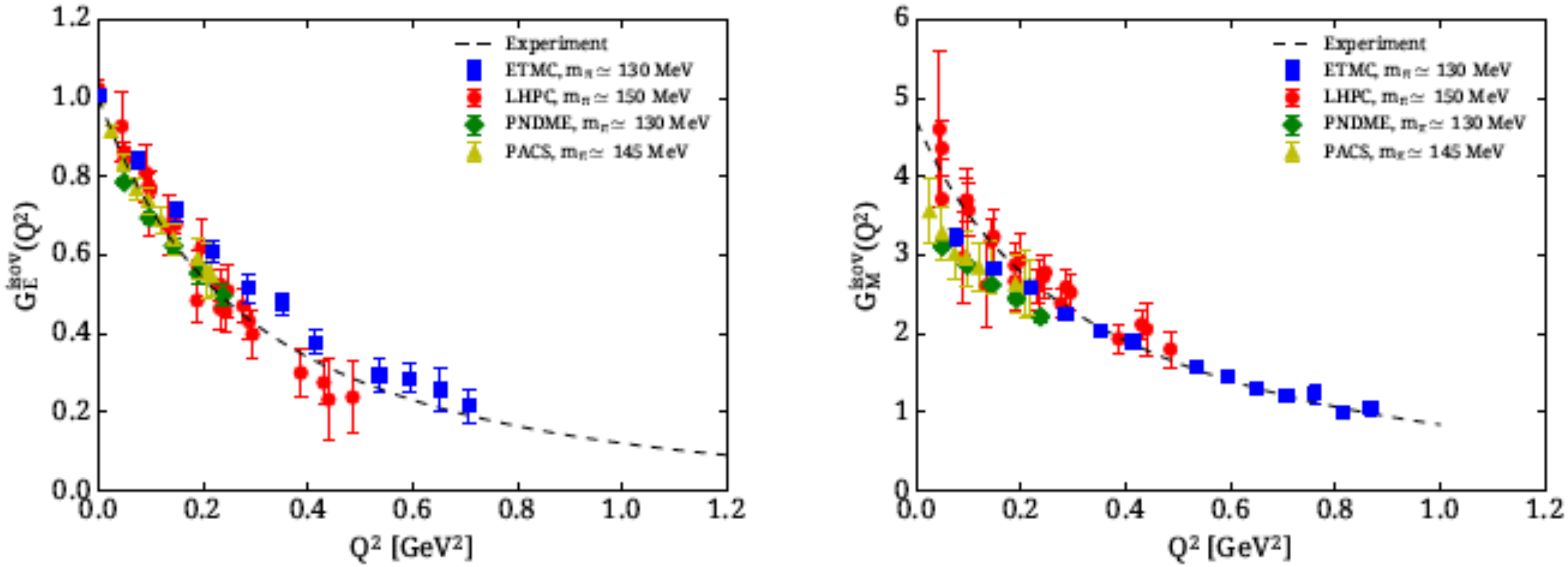
Proton charge radius



Proton charge radius

From Alexandrou @ CONF12

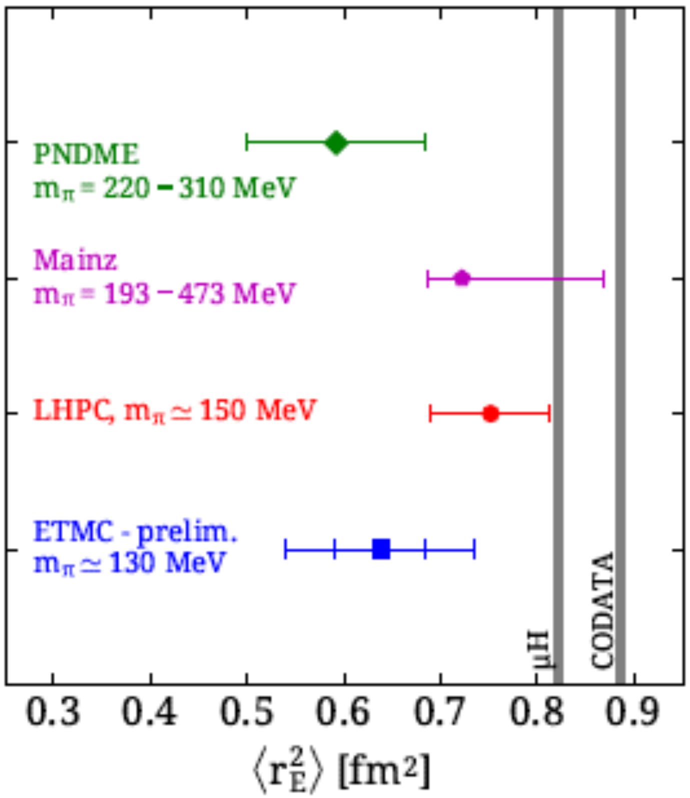
Isovector form factors

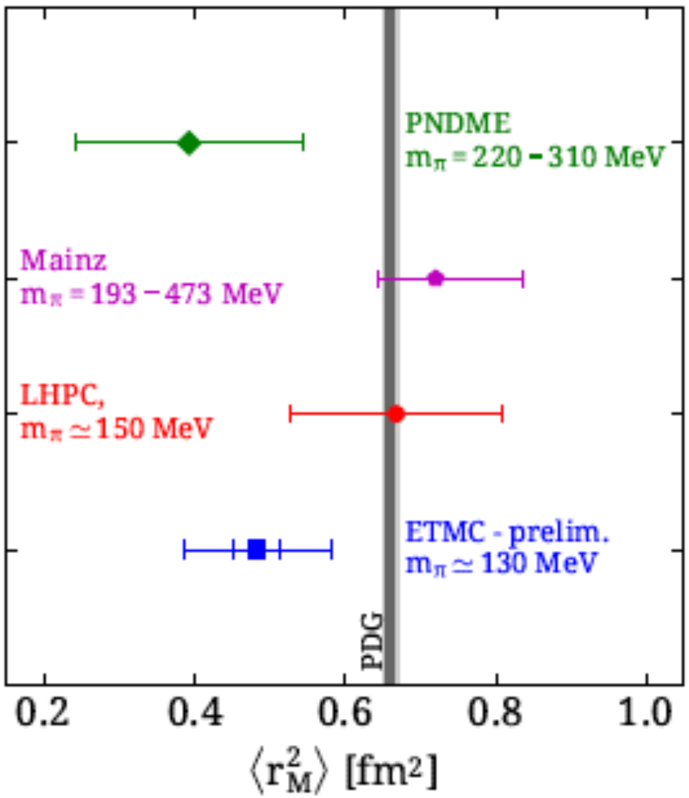


Proton charge radius

NOT precise enough…

From Alexandrou @ CONF12





Axial form factor

- Similar calculation

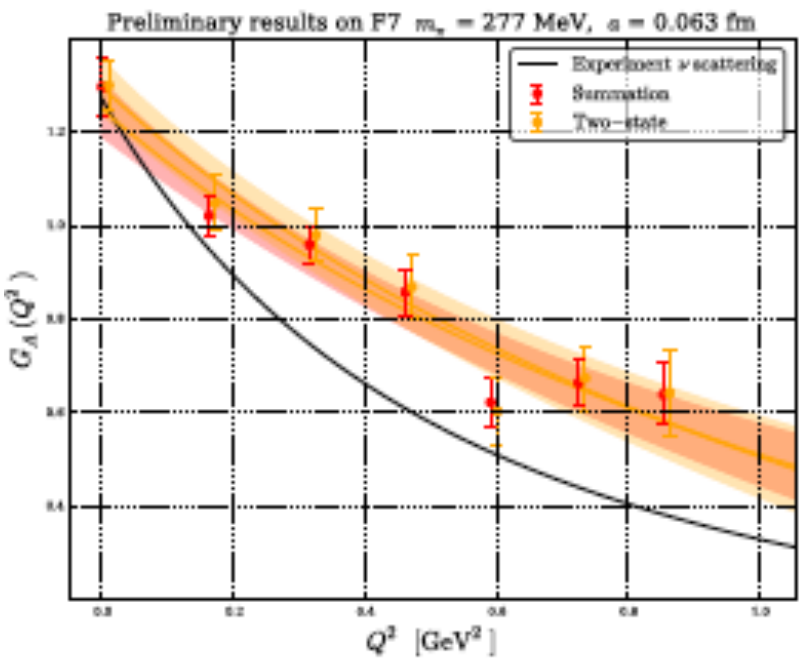
$$\langle N(p',s')|A_\mu^3|N(p,s)\rangle = \frac{i}{2}\left(\frac{m_N^2}{E_N(\mathbf{p}')E_N(\mathbf{p})}\right)^{1/2} \bar{u}(p',s')\left[\gamma_\mu\gamma_5 G_A(q^2) + \frac{q_\mu\gamma_5}{2m_N}G_P(q^2)\right]u(p,s)$$

- Traditionally, use the dipole form to fit the exp data

$$G_A(Q^2) = \frac{g_A}{(1+\frac{Q^2}{M_A{}^2})^2} \qquad \langle r_A^2\rangle = \frac{12}{M_A{}^2}$$

  - $M_A \sim 1$ GeV.

Axial form factor

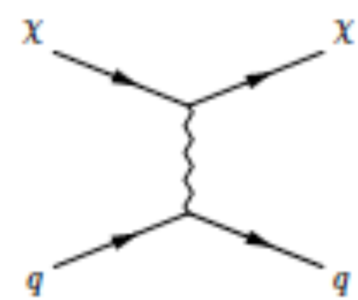


Mainz (2016)

Sigma term

$$\sigma_q = m_q(\langle N|\bar{q}q|N\rangle - \langle 0|\bar{q}q|0\rangle)$$

- Relevant to the dark matter detection, if DM couples to the scalar current.



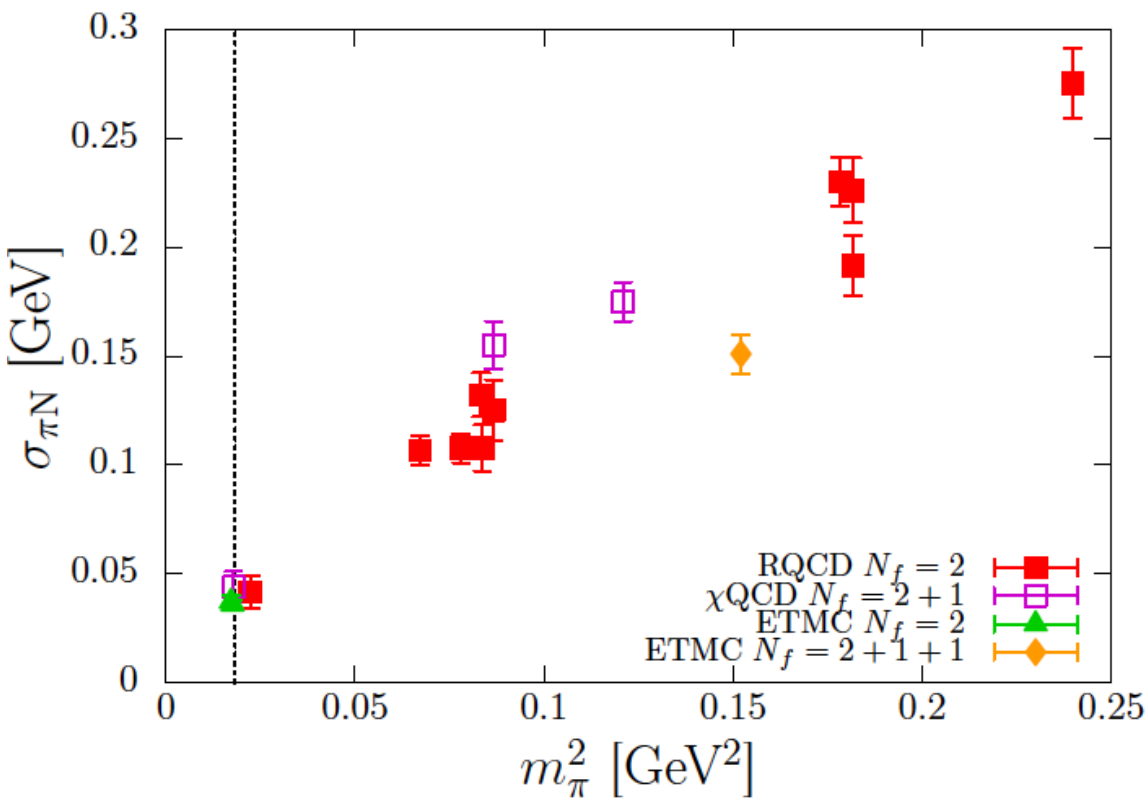
- Feynman-Hellmann theorem

$$m_q\langle N|\bar{q}q|N\rangle = m_q\frac{\partial}{\partial m_q}m_N$$

Sigma term

Direct

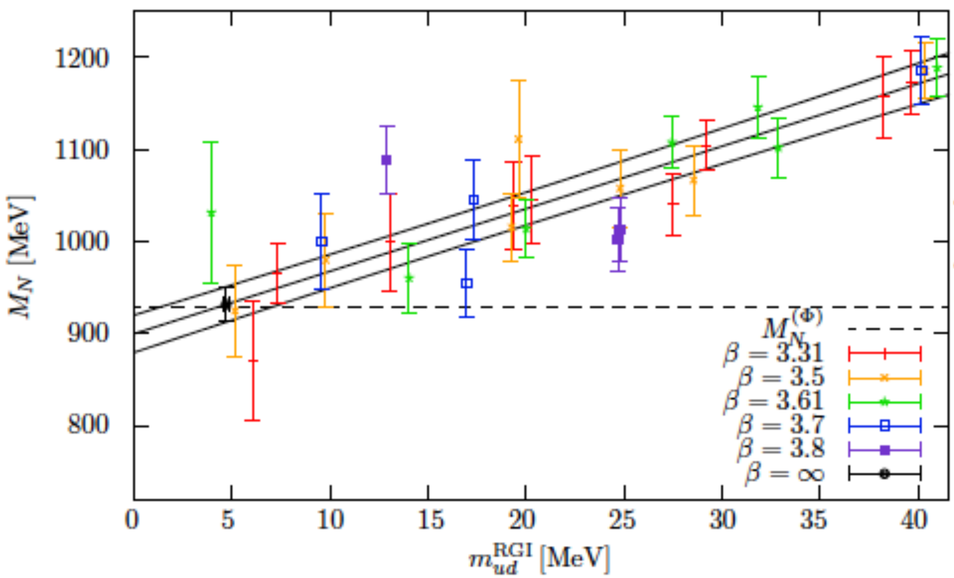
From Collins @ Lattice 2016

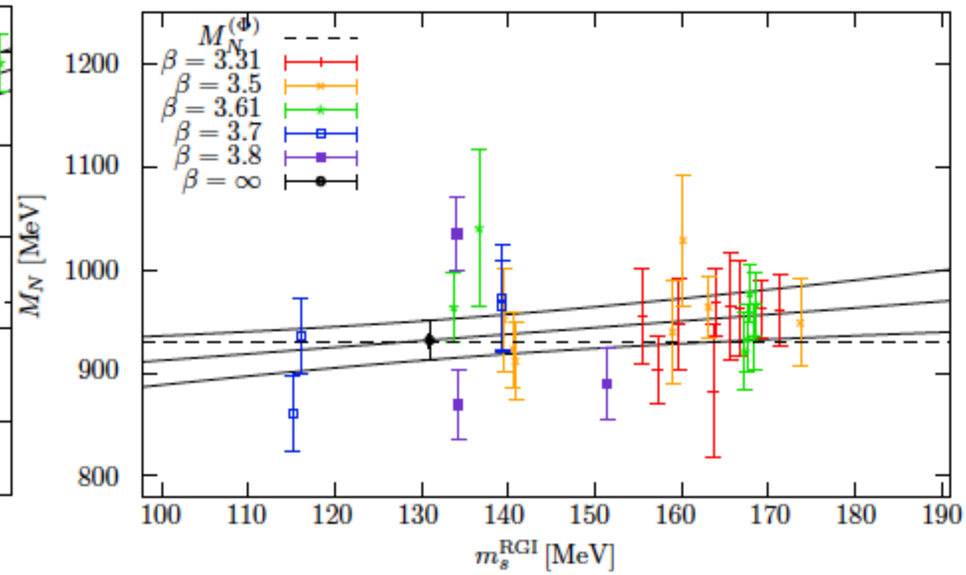


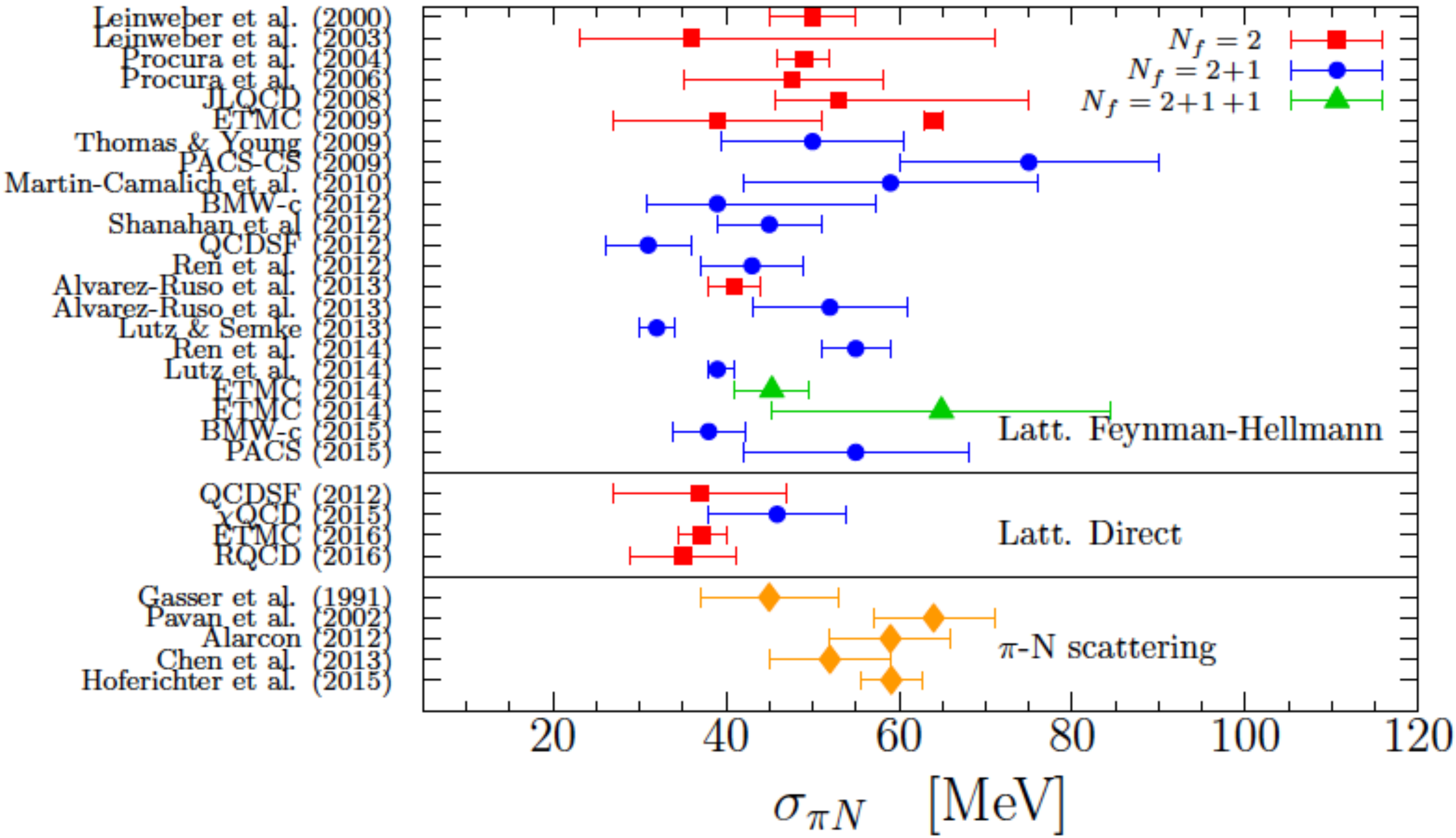
Sigma term

Feynman-Hellmann

BMW (2016)

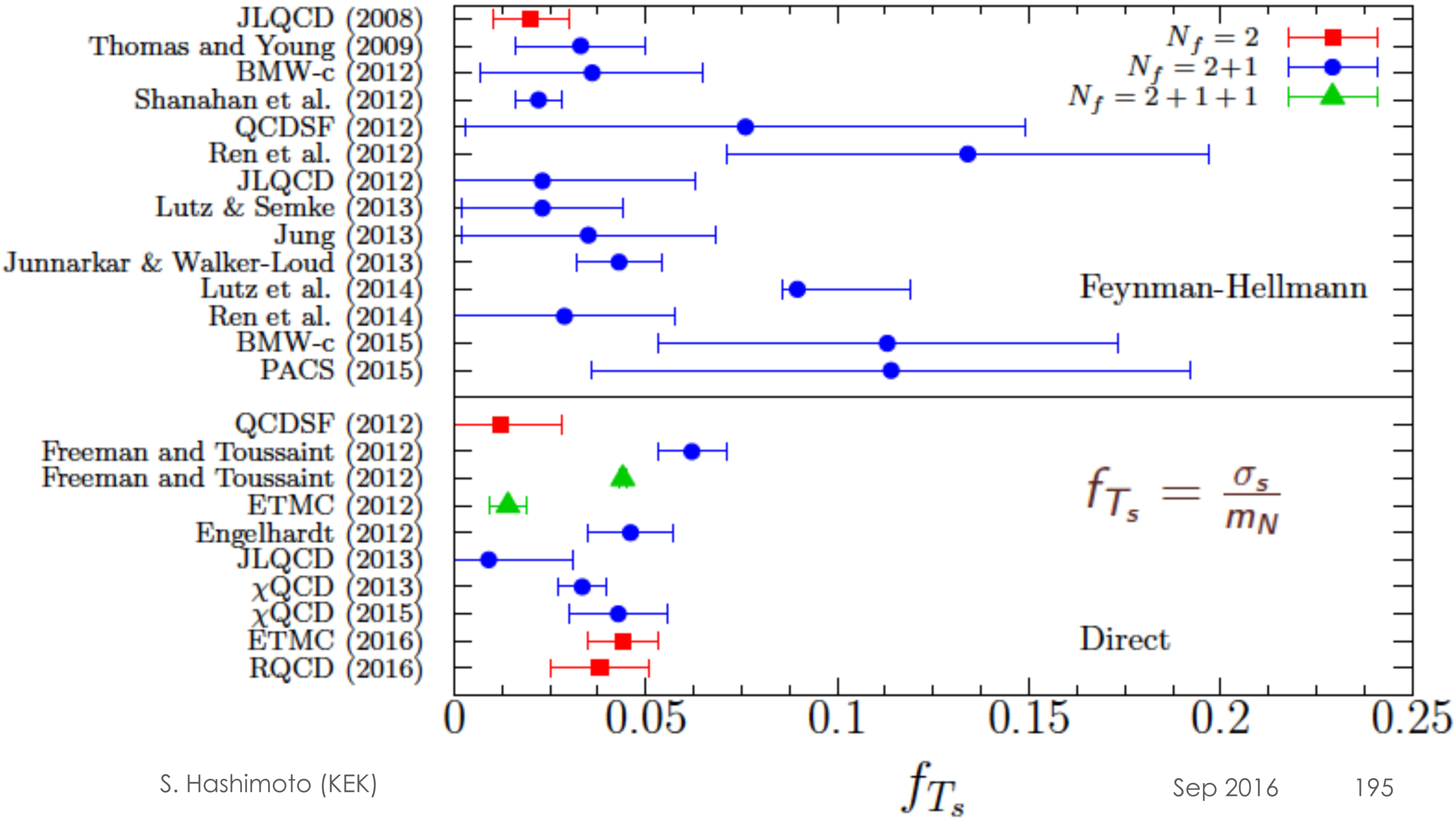






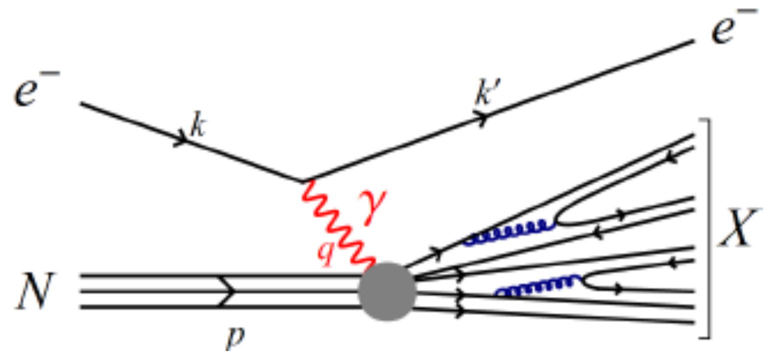
Leinweber et al. (2000)
Leinweber et al. (2003)
Procura et al. (2004)
Procura et al. (2006)
JLQCD (2008)
ETMC (2009)
Thomas & Young (2009)
PACS-CS (2009)
Martin-Camalich et al. (2010)
BMW-c (2012)
Shanahan et al (2012)
QCDSF (2012)
Ren et al. (2012)
Alvarez-Ruso et al. (2013)
Alvarez-Ruso et al. (2013)
Lutz & Semke (2013)
Ren et al. (2014)
Lutz et al. (2014)
ETMC (2014)
ETMC (2014)
BMW-c (2015)
PACS (2015)
QCDSF (2012)
χQCD (2015)
ETMC (2016)
RQCD (2016)
Gasser et al. (1991)
Pavan et al. (2002)
Alarcon (2012)
Chen et al. (2013)
Hoferichter et al. (2015)
$N_f = 2$
$N_f = 2+1$
$N_f = 2+1+1$
Latt. Feynman-Hellmann
Latt. Direct
π-N scattering
20
40
60
80
100
120
$\sigma_{\pi N}$ [MeV]

Strange quark content



Structure functions

Deep Inelastic Scattering (DIS)



cross section

$$\sigma \sim L^{\mu\nu} W_{\mu\nu},$$

$$W_{\mu\nu} = i \int d^4x e^{iqx} \langle N | T\{J^\mu(x), J^\nu(0)\} | N \rangle$$

structure functinos

$$W^{\{\mu\nu\}}(x,Q^2) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x,Q^2) + \left(p^\mu - \frac{\nu}{q^2} q^\mu\right)\left(p^\nu - \frac{\nu}{q^2} q^\nu\right) \frac{F_2(x,Q^2)}{\nu}$$

Structure functions

Moments:

$$2\int dx\, x^{n-1} F_1(x,Q^2) = \sum_{q=u,d} c_{1,n}^{(q)}(\mu^2/Q^2, g(\mu)) \langle x^n \rangle_q$$

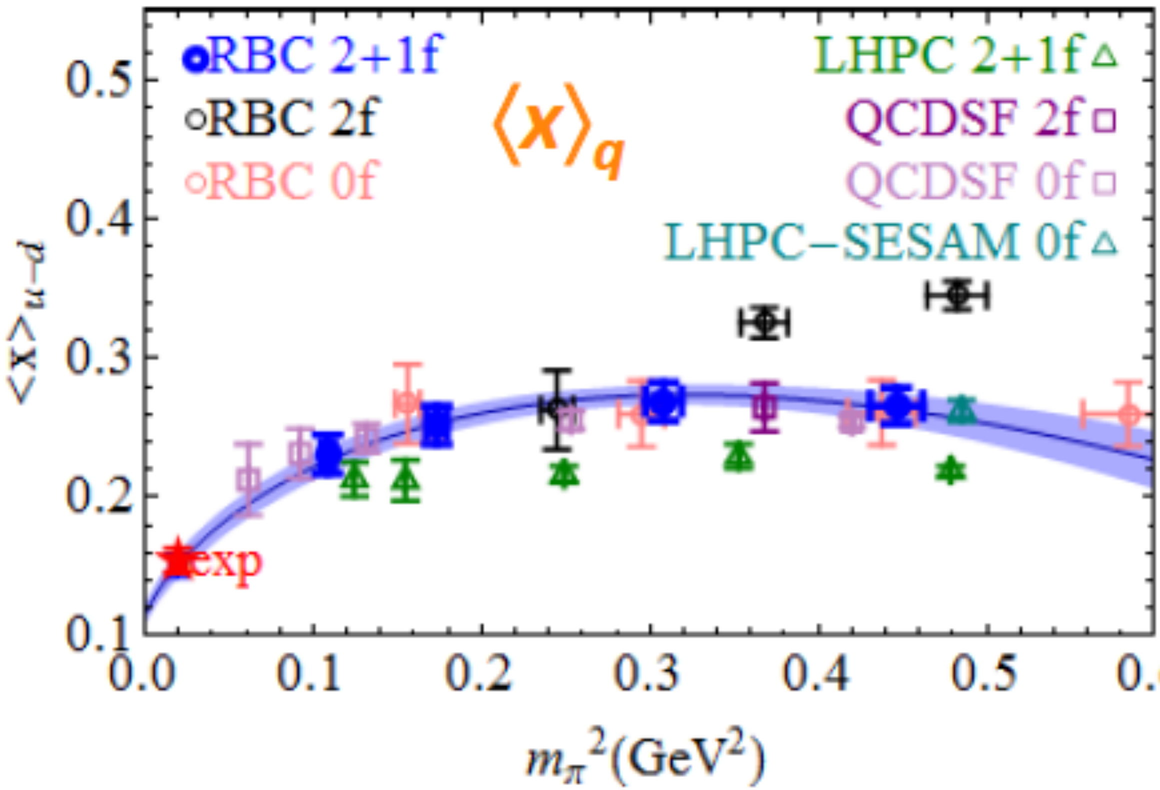
$$\int dx\, x^{n-2} F_2(x,Q^2) = \sum_{q=u,d} c_{2,n}^{(q)}(\mu^2/Q^2, g(\mu)) \langle x^n \rangle_q$$

can be written in terms of matrix elements of

$$\mathcal{O}^q_{\mu_1\mu_2\cdots\mu_n} = \left(\frac{i}{2}\right)^{n-1} \bar{q}\gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n}\, q - trace$$

Structure functions

From Lin (2010)



Summary

History

<u>40 years of LQCD</u>

1974 K. Wilson's seminal paper

1980s Early numerical simulations. Sea quarks were ignored in most computations.

1990s A lot of technical improvements, still quenched.

2000s Dynamical quarks are attempted, still too heavy.

2010~ Low-lying hadron spectrum (+ some other "easy") quantities are okay.

<u>And, next 10 years</u>

2016~ LQCD as a tool of precise computation of the Standard Model

Challenges, not covered

- QED
  - Needs to be included to go below 1%.
  - Non-trivial due to long-range force. Some attempts already exist. No generally applicable method.
- Multi-body systems (scatterings, decays, exotics)
  - Needs dedicated theoretical framework to connect Euclidean correlation function to the physical quantities. Exists for two-body.
  - A lot of attempts. Works for simplest system (I=2 $\pi\pi$).
- Topological quantities
  - Non-trivial to define the topological charge on the lattice. Easier on sufficiently fine lattices, but then the topology freezes.

Challenges, not covered

- Finite temperature
  - Phase transition is not easy to identify on finite volumes.
  - A lot of studies have been done. Consensus is a "crossover" for 2+1-flavor QCD.
  - Non-trivial when the topology is relevant.
- Finite density
  - Sign problem: MC doesn't work.
  - A lot of attempts without full success.
  - Related to a problem of statistical noise.

Summary

Not a comprehensive lecture. Some feeling about QCD and its numerical simulation.

- QCD is simple, but non-linear.
- Rich structure
  - Asymptotic freedom
  - Confinement
  - Chiral symmetry breaking

- Lattice QCD: Non-perturbative calculation of QCD has become feasible. Now, a precision physics.

事前質問

格子QCD

- 格子QCDの計算がどこまでできているか
- 格子計算のインプットとしてはどんなものが必要か
- AdS/CFT対応で重力理論とゲージ場が繋がるが、その辺と格子QCDの関係や成果を教えてほしい。
- 格子QCDの素粒子、原子核、宇宙物理それぞれでの位置付けと成果を知りたい。
- 格子QCD計算の次世代計画(コスモシミュレータ計画？)とそれによって新たにわかる物理

事前質問

実験とQCD

- 格子QCDがLHC物理でのハドロン現象に対して役に立っているか。また、立っているならその詳細を知りたい。
- g-2測定のハドロンループの計算誤差の入り方
- 0nbb の核行列要素（どの原子核が有利かってのはどれくらい妥当なのか？)
- 物性物理との対応や応用もあれば教えてほしい。