電磁場中 の 粒子の運動方程式

$$\frac{d\vec{p}}{dt} = q\vec{E} + ec\vec{\beta} \times \vec{B} \qquad \vec{\beta} = \left(\frac{v_x}{c}, \frac{v_y}{c}, \frac{v_z}{c}\right)$$

ラグランジアン $\quad \Phi, \vec{A}$

$$L = -mc^2\sqrt{1-\beta^2} - e\Phi + ec\vec{\beta}\vec{A}$$

オイラーラグランジュ方程式

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) - \frac{\partial L}{\partial \vec{x}} = 0$$

$$\beta^2 = \sqrt{\frac{v_x^2}{c^2} + \frac{v_y^2}{c^2} + \frac{v_z^2}{c^2}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{r}}\right) = \frac{d}{dt}\left(-\frac{1}{\cancel{2}} m \cancel{c^2} \underbrace{\left(1-\frac{\vec{v}^2}{c^2}\right)}_{\gamma}{}^{-\frac{1}{2}}\left(-\frac{\cancel{2}\vec{v}}{\cancel{c^2}}\right) + e\vec{A}\right)$$

$$= \frac{d}{dt}\left(\underbrace{m\gamma\vec{\beta}c}_{\longrightarrow\vec{p}} + e\vec{A}\right) = \frac{d}{dt}\vec{p} + e\frac{\partial \vec{A}}{\partial t} + e\vec{v}\cdot\vec{A}$$

$$\frac{d}{dt}\vec{A} = \frac{\partial \vec{A}}{\partial t} + \vec{v}\cdot\vec{A}$$

$$\left(\frac{\partial L}{\partial \vec{x}}\right)_x = -e\frac{\partial \Phi}{\partial x} + ec\vec{\beta}\cdot\frac{\partial \vec{A}}{\partial x}$$

$$= -e\frac{\partial \Phi}{\partial x} + ec\,\beta_x\left(\frac{\partial A_x}{\partial x} + \beta_y\frac{\partial A_y}{\partial x} + \beta_z\frac{\partial A_z}{\partial z}\right)$$

$$\left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \vec{v}}\right)_x - \left(\frac{\partial L}{\partial \vec{x}}\right)_x = \frac{d}{dt}\vec{p}_x + e\frac{\partial A_x}{\partial t} + e\frac{\partial \Phi}{\partial x}$$

$$- e\left(v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right)$$

$$\vec{B} = \mathrm{rot}\,\vec{A}$$

$$\vec{E} = -\mathrm{grad}\,\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\downarrow$$

$$\frac{d\vec{p}}{dt} = q\vec{E} + qc\vec{\beta} \times \vec{B}$$

Lagrangian is OK

$$\frac{\partial L}{\partial \vec{v}} = \vec{p}$$

ルジャンドル変換

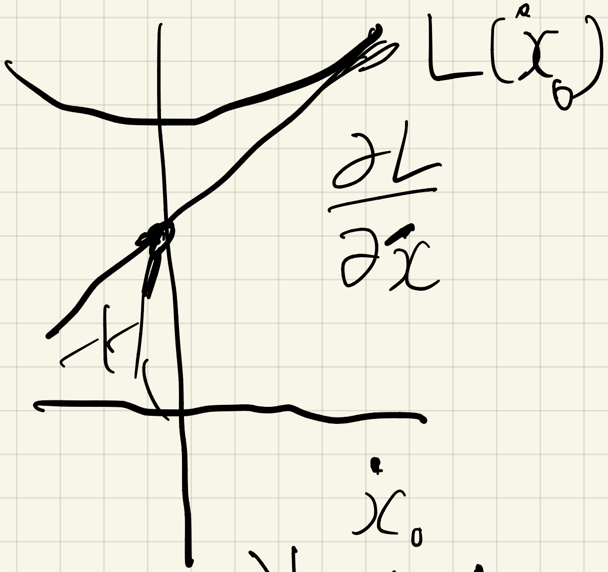
$$H(\vec{x}, \vec{p}, t)$$

$$= \vec{p}\cdot\vec{v} - L$$

$\vec{v} = \vec{v}(\vec{x}, \vec{p})$ とできる場合に限る

$$\frac{\partial(\vec{x}, \vec{p})}{\partial(\vec{x}, \vec{v})} \neq 0$$

次ページ



$$W = \frac{\partial L}{\partial \dot{x}}(\dot{x} - \dot{x}_0) + L(\dot{x}_0)$$

$$-H = -\frac{\partial L}{\partial \dot{x}}\dot{x}_0 + L(\dot{x}_0)$$

$$H = \frac{\partial L}{\partial \dot{x}}\dot{x}_0 + L(\dot{x}_0)$$

覚え方.

$\frac{\partial(\vec{x},\vec{p})}{\partial(\vec{x},\vec{v})} \neq 0$ の証明

$$\vec{P} = m\gamma\vec{\beta}c + e\vec{A}$$

$$\frac{\partial P_x}{\partial v_x} = \frac{\partial}{\partial v_x} m\left(1-\frac{v_x^2+v_y^2+v_z^2}{c^2}\right)^{-\frac{1}{2}}\frac{v_x}{c}\times c$$

$$= -\frac{1}{2}m\left(1-\frac{v_x^2+v_y^2+v_z^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v_x}{c^2}\right)v_x + m\gamma$$

$$\doteq m\beta_x^2\gamma^3 + m\gamma$$

$$\frac{\partial P_x}{\partial v_y} = m\beta_y\beta_x\gamma^3$$

$$\frac{\partial P_x}{\partial v_y} = m\beta_x\beta_z\gamma^3$$

$\frac{\partial(\vec{x},\vec{p})}{\partial(\vec{x},\vec{v})} \neq 0$ の証明

$$\det\begin{pmatrix} I & 0 \\ B & \left(\frac{\partial p_i}{\partial v_j}\right) \end{pmatrix} = \det\left(\frac{\partial P_i}{\partial v_j}\right) = \begin{pmatrix} \beta_x^2\gamma^2+1 & \beta_x\beta_y\gamma^2 & \beta_x\beta_z\gamma^2 \\ \beta_x\beta_y\gamma^2 & \beta_y^2\gamma^2+1 & \beta_y\beta_z\gamma^2 \\ \beta_x\beta_z\gamma^2 & \beta_y\beta_z\gamma^2 & \beta_z^2\gamma^2+1 \end{pmatrix}$$

$$(\beta_x^2\gamma^2+1)\left\{(\beta_y^2\gamma^2+1)(\beta_z^2\gamma^2+1) - \beta_y^2\beta_z^2\gamma^4\right\}$$

$$-\beta_x\beta_y\gamma^2\left\{\beta_x\beta_y(\cancel{\beta_z^2\gamma^2}+1) - \cancel{\beta_y\beta_z^2\beta_x\gamma^4}\right\}$$

$$+\beta_x\beta_z\gamma^2\left\{\cancel{\beta_x\beta_y^2\beta_z\gamma^4} - \cancel{\beta_x\beta_z\beta_y^2\gamma^4} - \beta_x\beta_z\gamma^2\right\}$$

$$(\beta_x^2\gamma^2+1)(\beta_y^2\gamma^2+\beta_z^2\gamma^2+1)$$

$$-\beta_x^2\beta_y^2\gamma^4$$

$$-\beta_x^2\beta_z^2\gamma^4$$

$$= \beta_x^2\gamma^2+\beta_y^2\gamma^2+\beta_z^2\gamma^2+1$$

$$>0$$

$$\vec{P} = \frac{\partial L}{\partial \vec{v}} = m\gamma\vec{\beta}c + e\vec{A} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$H = \vec{P}\cdot c\vec{\beta} - L$$

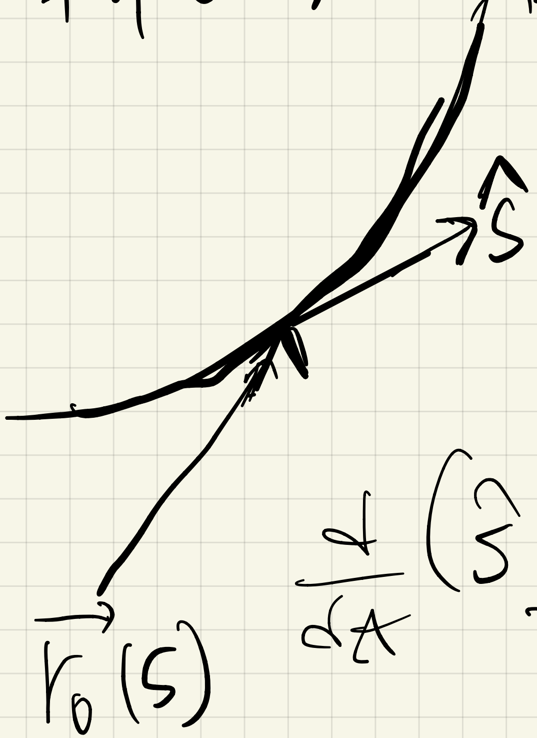
$$= c\vec{p}\cdot\vec{\beta} + mc^2\frac{1}{\gamma} + e\Phi - ec\vec{\beta}\cdot\vec{A}$$

$$= m\gamma\beta^2c^2 + mc^2\frac{1}{\gamma} + e\Phi$$

$$= m\gamma c^2 + e\Phi$$

$$\underline{\underline{= \sqrt{c^2(\vec{P}-e\vec{A})^2 + m^2c^4} + e\Phi}}$$

新しい座標 Frenel serret 座標



$$\hat{s}(s) = \frac{d r_0(s)}{ds} \qquad \hat{x}(s) = -\rho(s)\cdot\frac{d\hat{s}(x)}{ds}$$

$$\hat{y}(s) = \hat{s}(s)\times\hat{x}(s)$$

$\hat{x}(s)$

$$\frac{d}{dt}\left(\underbrace{\hat{s}(s)\cdot\hat{s}(s)}_{\text{const}}\right) = 0 \qquad 2\dot{\hat{s}}(s)\cdot s(s) = 0$$

$$\hat{x}(s)\cdot s(s) = 0$$

→ $\hat{x}(s)\cdot s(s)$ は直交する → $\hat{x}, \hat{y}, \hat{s}$ は互に直交

$$\frac{d}{dt}\left(\underbrace{\hat{x}(s)\cdot\hat{x}(s)}_{\text{const}}\right) = 2\hat{x}(s)\cdot\dot{\hat{x}}(s) = 0$$

一般的に $\dot{\hat{x}}(s) = a(s)\cdot s(s) + b(s)\cdot y(s)$ とできる

$$\frac{d}{dt}\Big(\underbrace{\hat{s}(s)\cdot\hat{x}(s)}_{\hookrightarrow 0}\Big) = 0 \qquad \dot{\hat{s}}(s)\,\hat{x}(s) + \hat{s}(s)\,\dot{\hat{x}}(s) = 0$$

$$\dot{\hat{s}}(s) = -\frac{\hat{x}(s)}{\rho(s)} \text{ なので } \qquad \hat{s}(s)\cdot\dot{\hat{x}}(s) = \frac{1}{\rho(s)}$$

$$\dot{\hat{x}}(s) = \frac{1}{\rho(s)}\hat{s}(s) + b(s)\cdot y(s)$$

$$\frac{d}{dt}\Big(\underbrace{\hat{y}(s)\cdot\hat{x}(s)}_{\hookrightarrow 0}\Big) = \underbrace{\dot{\hat{x}}(s)\cdot\hat{y}(s)}_{\hookrightarrow b(s)} + \hat{x}(s)\cdot\dot{\hat{y}}(s)$$

$$\hat{x}(s)\cdot\dot{\hat{y}}(s) = -b(s)$$

$\hat{x}(s)$

$$\frac{d}{dt}\Big(\underbrace{\hat{y}(s)\cdot\hat{s}(s)}_{\hookrightarrow 0}\Big) = \dot{\hat{y}}(s)\,\hat{s}(s) + \underbrace{\hat{y}(s)\,\dot{\hat{s}}(s)}_{\hookrightarrow 0} \to \dot{\hat{y}}(s)\cdot s(x) = 0$$

$$\frac{d}{dt}\left(\underbrace{\hat{y}(s)\cdot\hat{y}(s)}_{\text{const}}\right) = 2\hat{y}(s)\dot{\hat{y}}(s) = 0$$

$$\dot{y}(s) = -b(s)\cdot x(s)$$

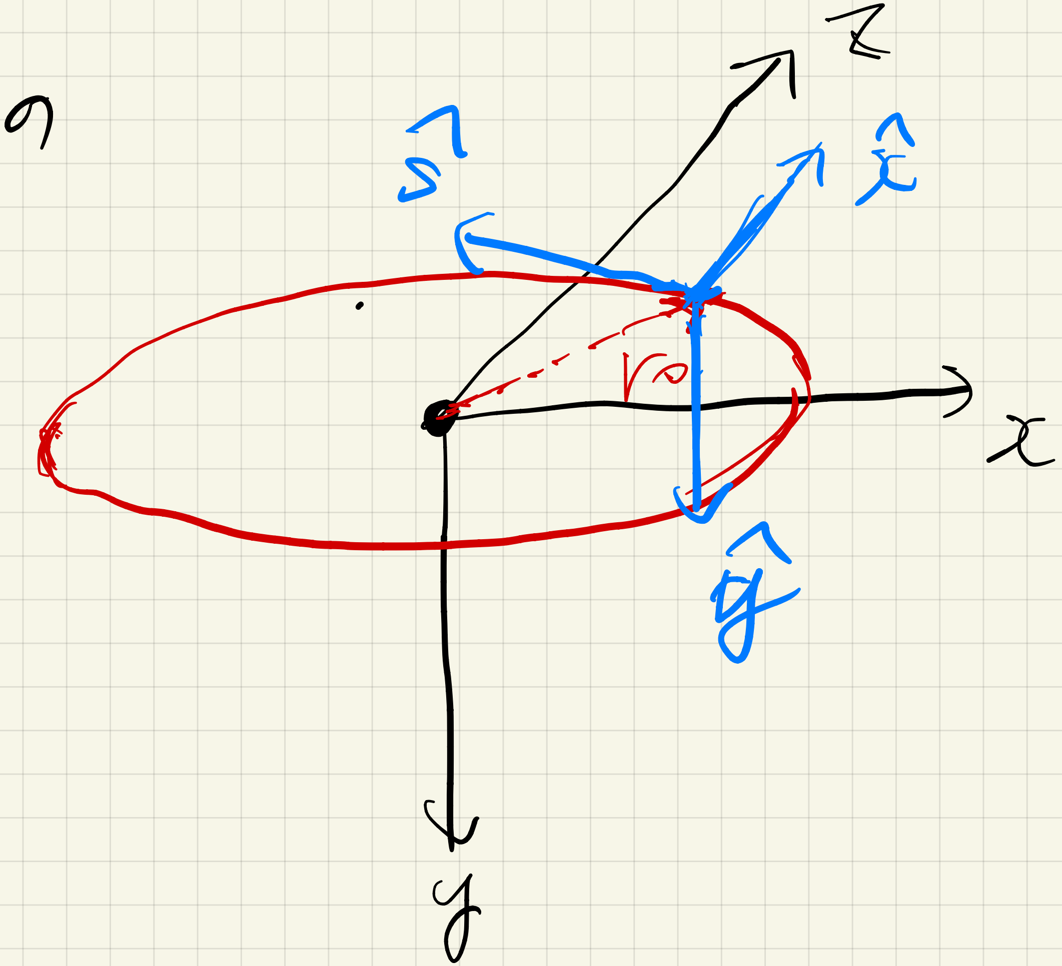
ここで $b(s) = 0$ ととると $\dot{\hat{y}}(s) = 0$

$\hat{y}(s) = \text{const}$ とできる 常に同じ方向

$$\Updownarrow$$

$\hat{s}(s)$ と $\hat{x}(s)$ は同一平面上

本講義の座標系



$\vec{r} = \vec{r_0} + x\hat{x}(s) + y\hat{y}(s)$

x : Horizontal 方向

y : Vertical 方向

$$\vec{r} = \vec{r_0} + x\hat{x}(s) + y\hat{y}(s)$$

これでハミルトニアンを考えたい。

$qp \rightarrow PQ$ 正準変換

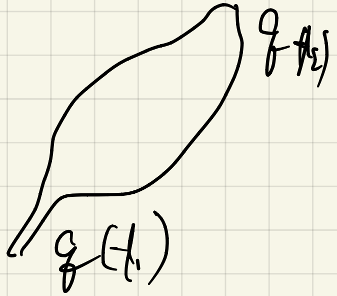
$F_3(p, Q, t)$ 母関数

$$\vec{P} = -\frac{\partial F_3}{\partial Q} \qquad \vec{q} = -\frac{\partial F_3}{\partial \vec{p}}$$

$$H' = H + \frac{\partial F_3}{\partial t}$$

正準変換

〈最小作用の原理〉



$$\delta\left[\int_{t_1}^{t_2}\underbrace{(p\dot{q}-H)}_{\hookrightarrow L(q,\dot{q},t)}dt\right]=0$$ だった

$$\int p\dot{q}\,dt=-\int\dot{p}q\,dt+\underbrace{\Big[p\cdot q\Big]_{t_1}^{t_2}}_{\circlearrowleft}$$

行き方によらない

$$\delta\int_{t_1}^{t_2}(-\dot{p}q-H)\,dt=0$$ でもよい

$$p\dot{q} - H = P\dot{Q} - H' + \frac{dF_1}{dt} \to p\,dq - H\,dt = P\,dQ - H'\,dt + dF_1$$

$$-\dot{p}q - H = P\dot{Q} - H' + \frac{dF_3}{dt}$$

$$p\dot{q} - H = -\dot{P}Q - H' + \frac{dF_2}{dt}$$

$$-\dot{p}q - H = -\dot{P}Q - H' + \frac{dF_4}{dt}$$

$$H' = H + \frac{\partial F_1}{\partial t}, \quad \frac{\partial F_1}{\partial q} = p \quad \frac{\partial F_1}{\partial Q} = -P \quad //$$

$$H' = H + \frac{\partial F_3}{\partial t}, \quad \frac{\partial F_3}{\partial p} = -q \quad \frac{\partial F_3}{\partial Q} = -P$$

$$H' = H + \frac{\partial F_2}{\partial t}, \quad \frac{\partial F_2}{\partial q} = p \quad \frac{\partial F_2}{\partial P} = Q$$

$$H' = H + \frac{\partial F_4}{\partial t}, \quad \frac{\partial F_4}{\partial p} = -q \quad \frac{\partial F_4}{\partial P} = Q$$

$$F_3 = -\vec{P}_{old} \cdot \left(\vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s) \right)$$

$$(P_{New})_s = -\frac{\partial F_3}{\partial s} = \vec{P}_{old}\left(\hat{s}(s) + \frac{x}{\rho(s)}\hat{s}(s) \right)$$

$$= \vec{P}_{old}\left(1 + \frac{x}{\rho}\right) \cdot \hat{s}(s)$$

$$\hat{x}'(s) = \frac{1}{\rho(s)}\hat{s}(s)$$

$$(P_{New})_x = \vec{P}_{old} \cdot \hat{x}(s)$$

$$(P_{New})_y = \vec{P}_{old} \cdot \hat{y}(s)$$

$A_x = \vec{A}\cdot\hat{x}$ $A_y = \vec{A}\cdot\hat{y}$ $A_s = \left(1+\frac{x}{\rho}\right)\vec{A}\cdot\hat{s}$

$$H = c\left((p_x - eA_x)^2 + (p_y - eA_y)^2 + \frac{(p_s - eA_s)^2}{\left(1+\frac{x}{\rho}\right)^2} + m^2c^2\right)^{\frac{1}{2}}$$

$+ e\Phi$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p_x} \\ \frac{dp_x}{dt} &= -\frac{\partial H}{\partial x} \end{aligned}}$$

独立変数が時間の関数

$\longrightarrow$ s 曲線に沿ったキョリ

$t \longleftrightarrow s$

$$S = \int \left(\underbrace{p_x \dot{x} + p_y \dot{y} + p_s \dot{s}}_{L} - H \right) dt$$

$$H = p \cdot \dot{q} - L$$

$$\delta S = \int \left(\delta p_x \dot{x} + p_x \delta \dot{x} + \delta p_y \dot{y} + p_y \delta \dot{y} + \delta p_s \dot{s} + p_s \delta \dot{s} - \delta H \right) dt$$

$$\delta H = \frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial y} \delta y + \frac{\partial H}{\partial s} \delta s + \frac{\partial H}{\partial p_x} \delta p_x + \frac{\partial H}{\partial p_y} \delta p_y + \frac{\partial H}{\partial p_s} \delta p_s$$

$$\int (p_x \delta \dot{q})\, dt = \underbrace{\left[p_x \delta x \right]_{t_0}^{t}}_{\hookrightarrow 0} - \int \dot{p}_x\, \delta x\, dt$$

$$0 = \delta S = \int \delta p_x \left(\dot{x} - \frac{\partial H}{\partial p_x} \right) + \delta x \left(- \dot{p} - \frac{\partial H}{\partial x} \right) = 0$$

ここで $dt = \frac{dt}{ds} ds$ を代入し プライム′をs微分とする

$$S = \int \left(p_x x' + p_y y' + (-H) \underbrace{\frac{dt}{ds}}_{\hookrightarrow t'} + (-p_s) \right) ds$$

すると $(x, y, t, p_x, p_y, -H)$ も正準変数

$-p_s$ を Hamiltonian, s を独立変数とする

$$H_0 = -p_s$$

粒子の運動エネルギー

$$= \left[\left\{\left(\frac{H - e\Phi}{c}\right)^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2 - m^2c^2\right\}^{\frac{1}{2}} \times \left(1 - \frac{x}{\rho}\right) + eA_s\right] \times (-1)$$ 〃

$\sigma = s - \beta_0 c t$ ：正準変数

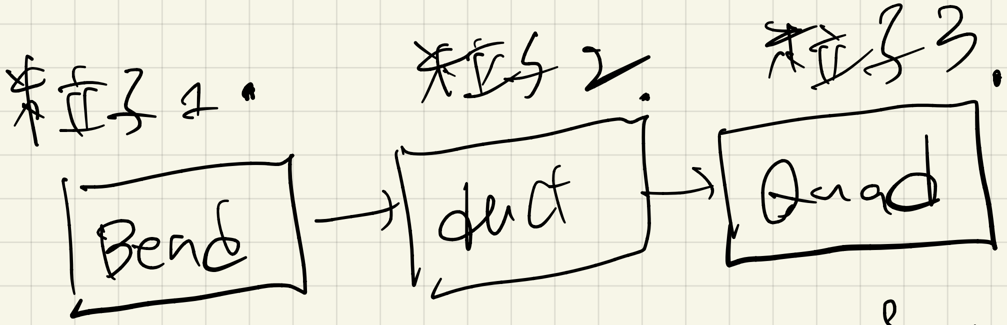
$$F_3(-H, \sigma, s) = -(-H)\left(\frac{s - \sigma}{\beta_0 c}\right)$$

$$-\frac{\partial F_3}{\partial(-H)} = t \qquad -\frac{\partial F_3}{\partial \sigma} = p_\sigma = \frac{H}{\beta_0 c} \quad 〃 \qquad \frac{\partial F_3}{\partial s} = \frac{H}{\beta_0 c}$$

少なくとも s をとると（手引）計算で便利

変数 t だと

$t = t_0$ の時

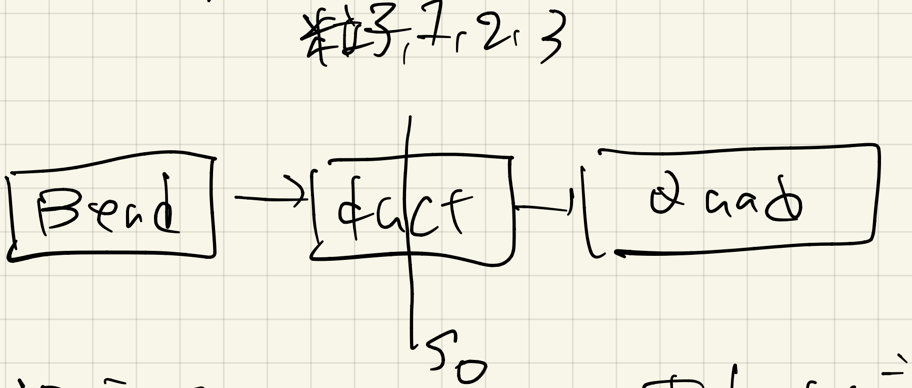


一般的に ある t_0 のとき 別々のコンポーネントにいる

粒子毎に違う関数を呼び出す。

変数 s だと。

$s = s_0$



同じ関数を全粒子にアプライすればよい

* 最近気づいた所、バンチ（粒子のかたまり）の瞬間的な縦方向分布が得にくいというデメリットもある

$A_x = A_y = \Phi = 0 \quad \nearrow \frac{\psi}{B_0 c}$

$$H_1 = -\left(1+\frac{x}{\rho}\right)\left[p_0^2 p_\sigma^2 - p_x^2 - p_y^2 - m^2c^2\right)^{\frac{1}{2}}$$

$$-eA_s + p_\sigma$$

$$= p_\sigma - eA_s - \left(1+\frac{x}{\rho}\right)\left(p^2 - p_x^2 - p_y^2\right)^{\frac{1}{2}}$$

$\frac{p_x}{p_0} \to p_x \qquad p_0$: reference 粒子の運動量

$H_2 = \frac{H_1}{p_0} \qquad \frac{dp_x}{ds} = -\frac{\partial H_1}{\partial x} \qquad \frac{dx}{ds} = \frac{\partial H_1/p_0}{\partial p_x/p_0}$

$$H_2 = p_\sigma - e\frac{A_s}{p_0} - \left(1+\frac{x}{\rho}\right)\left\{(1+\delta)^2 - p_x^2 - p_y^2\right]^{\frac{1}{2}}$$

$$\delta = \frac{P - P_0}{P_0} = \frac{\Delta P}{P} \qquad p_\sigma = \frac{H}{\beta_0 p_0 c} \rightarrow p_\sigma = \frac{H - E_0}{\beta_0 p_0 c}$$

E_0 reference 粒子のエネルギー　　$H \rightarrow E$

$$\delta = \frac{p - p_0}{p_0} \qquad E = \beta_0 p_0 c \, p_\sigma + E_0$$

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad \delta = p_\sigma - \frac{1}{2}\frac{p_\sigma^2}{\gamma_0^2} \cdots\cdots$$

$$H_2 = p_\sigma - e\frac{A_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\left(1 + 2p_\sigma + \beta_0^2 p_\sigma^2 - p_x^2 - p_y^2\right)^{\frac{1}{2}}$$

$$H_2 = -\frac{eA_s}{p_0} + \frac{p_x^2+p_y^2}{2} + \frac{p_r^2}{2p_0^2} - \frac{1}{2}p_r(p_x^2+p_y^2) - (1+p_r)\frac{x}{\rho}$$

近似した Hamiltonian 〃（僕が項を選んでいる）

$$\vec{B} = \mathrm{rot}\vec{A} = \frac{1}{(1+\frac{x}{\rho})}\left(\frac{\partial A_s}{\partial y} - \cancel{\frac{\partial A_y}{\partial s}}\right)\hat{x} + \frac{1}{(1+\frac{x}{\rho})}\left(\cancel{\frac{\partial A_x}{\partial s}} - \frac{\partial A_s}{\partial x}\right)\hat{y} + \left(\cancel{\frac{\partial A_y}{\partial x}} - \cancel{\frac{\partial A_x}{\partial y}}\right)\hat{s}$$

$$(A_x, A_y, A_s) = (0, 0, A_s)$$

とおいた。

- さらに $p_r(p_x^2+p_y^2)$ の項は入れているのに同じオーダーの p_r^3 の項を入れていないのは本来おかしい。結局その近似がよいかどうかはそのようにしてつくった数値計算が実機マシンに使えるかどうかが重要。

- $\frac{x}{\rho}$ は $\frac{1}{\rho}$ が小さいので $\frac{x}{\rho}(p_y^2+p_y^2)$ は3乗のオーダーではなく4乗のオーダーとして捨てている

（栗本の解釈）